Section 16.6: Parametric Surfaces and Their Areas

A space curve is parametrized by the vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

A surface, $z = f(x, y)$, is parametrized by a vector function of two variables.
$\underline{\mathbf{r}(u, v)} = \underline{\langle x(u, v), y(u, v), z(u, v) \rangle}$ with (u, v) in region D.

<u>Useful Parametrizations</u>

- Surface given by $z = f(x, y)$.

Example: $z = x^2 + 4y^2$

$x = u$
$y = v$
$z = u^2 + 4v^2$

$r(u,v) = \langle u, v, u^2 + 4v^2 \rangle$

$x = x$
$y = y$
$z = x^2 + 4y^2$

$r(x,y) = \langle x, y, x^2 + 4y^2 \rangle$

Example: $x = z^2 + y^2$

$y = u$
$z = v$
$x = v^2 + u^2$

$r(u,v) = \langle v^2 + u^2, u, v \rangle$

$r(y,z) = \langle z^2 + y^2, y, z \rangle$

- Surface in cylindrical coordinates

Example: $x^2 + y^2 = 9$ for $0 \le z \le 2$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$x = 3\cos\theta$
$y = 3\sin\theta$
$z = z$

$r(z, \theta) = \langle 3\cos\theta, 3\sin\theta, z \rangle$

$0 \le z \le 2$
$0 \le \theta \le 2\pi$

- Surface in spherical coordinates

Example: $x^2 + y^2 + z^2 = 4$

$x = 2 \sin\phi \cos\theta$

$y = 2 \sin\phi \sin\theta$

$z = 2 \cos\phi$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

Example: Identify the surface with the given vector equation.

$$\mathbf{r}(u,v) = \langle \underbrace{u+2}_{x}, \underbrace{9+u^2+v^2+4u}_{y}, \underbrace{v}_{z} \rangle$$

$x = u + 2$

$x - 2 = u$

$y = 9 + u^2 + v^2 + 4u$

$z = v$

$y = 9 + (x-2)^2 + z^2 + 4(x-2)$

$= 9 + x^2 - 4x + 4 + z^2 + 4x - 8$

$y = x^2 + z^2 + 5$

parabaloid centered on y-axis opens is the positive y direction

Example: Find the tangent plane to the surface with parametric equations given below at the point (1, 4, 5).

$\mathbf{r}(u,v) = \langle u^3, v^2, u+2v \rangle$

$u^3 = 1 \qquad v^2 = 4 \qquad u + 2v = 5$

$u = 1$

$1 + 2v = 5$

$2v = 4$

$v = 2$

$r_u = \langle 3u^2, 0, 1 \rangle$

$r_v = \langle 0, 2v, 2 \rangle$

$r_u(1,2) = \langle 3, 0, 1 \rangle$

$r_v(1,2) = \langle 0, 4, 2 \rangle$

$r_u(1,2) \times r_v(1,2) = \dots = \langle -4, -6, 12 \rangle$

normal vector

$-4(x-1) - 6(y-4) + 12(z-5) = 0$

Note: In the special case the surface is defined by $z = f(x, y)$ and is parametrized by
$\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$. Then a normal vector is

$$n = \langle -f_x, -f_y, 1 \rangle$$

$$n = \langle f_x, f_y, -1 \rangle$$

Example: Find a normal vector for the surface defined as $x = f(y, z)$

$$\text{cross product} = \langle 1, -f_y, -f_z \rangle$$

$$= \langle -1, f_y, f_z \rangle$$

Definition: If a smooth parametric surface S is given by the equation $\mathbf{r}(u,v)$ and S is covered just once as (u,v) ranges throughout the parametric domain D, then the **surface area** of S is

$$A(S) = \iint_D dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

ds Line Integral $\Big\}$ dS Surface Integral

Example: Find the surface area for the surface given by $x = uv$, $y = u + v$, and $z = u - v$ where $u^2 + v^2 \le 1$

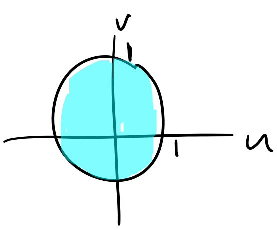
$r(u,v) = \langle uv, u+v, u-v \rangle$

$r_u = \langle v, 1, 1 \rangle$

$r_v = \langle u, 1, -1 \rangle$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} = \langle -2, -(-v-u), v-u \rangle$$

$$= \langle -2, v+u, v-u \rangle$$



polar

$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$$|r_u \times r_v| = \sqrt{(-2)^2 + (v+u)^2 + (v-u)^2}$$

$$= \sqrt{4 + v^2 + 2uv + u^2 + v^2 - 2uv + u^2}$$

$$= \sqrt{4 + 2u^2 + 2v^2} = \sqrt{4 + 2(u^2+v^2)}$$

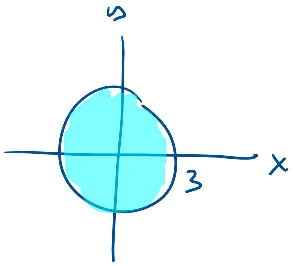
$$A(S) = \iint_S dS = \iint_D \sqrt{4 + 2(u^2+v^2)}\, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \sqrt{4+2r^2} \cdot r\, dr\, d\theta$$

$$= \dots = 2\pi\left(\sqrt{6} - \frac{4}{3}\right)$$

$z = 8 - 2x - 2y$

method 1

Example: Find the surface area for the part of the plane $2x + 2y + z = 8$ inside the cylinder $x^2 + y^2 = 9$.



$x = r\cos\theta$ $\qquad 0 \le r \le 3$

$y = r\sin\theta$ $\qquad 0 \le \theta \le 2\pi$

$z = 8 - 2r\cos\theta - 2r\sin\theta$

$r(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ 8 - 2r\cos\theta - 2r\sin\theta \rangle$

$r_r = \langle \cos\theta,\ \sin\theta,\ -2\cos\theta - 2\sin\theta \rangle$

$r_\theta = \langle -r\sin\theta,\ r\cos\theta,\ 2r\sin\theta - 2r\cos\theta \rangle$

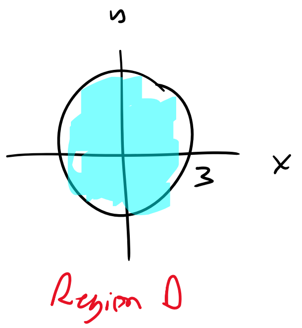
$r_r \times r_\theta = \dots = \langle 2r, 2r, r \rangle$

$|r_r \times r_\theta| = \sqrt{4r^2 + 4r^2 + r^2} = \sqrt{9r^2} = 3r$

$$SA = \iint_S 1\, dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} 3r\, dr\, d\theta = \dots = 27\pi$$

Example: Find the surface area for the part of the plane $2x+2y+z=8$ inside the cylinder $x^2+y^2=9$.

method 2)



$$x = x$$
$$y = y$$
$$f(x,y) = z = 8-2x-2y$$

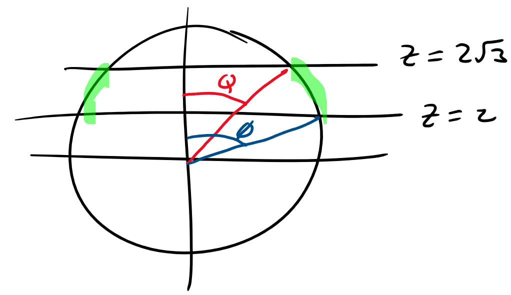
$$r(x,y) = \langle x,\ y,\ 8-2x-2y \rangle$$

$$\text{cross product} = \langle \underbrace{-(-2)}_{-f_x},\ \underbrace{-(-2)}_{-f_y},\ \underbrace{1}_{1} \rangle$$
$$= \langle 2,\ 2,\ 1 \rangle$$

$$|\text{cross product}| = \sqrt{2^2+2^2+1} = \sqrt{9} = 3$$

$$SA = \iint_S dS = \underbrace{\iint_D 3\, dA}_{3\cdot\underbrace{\pi(3)^2}_{\text{Area of } D} = 27\pi} = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} 3r\, dr\, d\theta = \dots = 27\pi$$

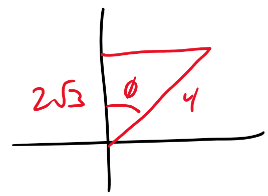
Example: Find the surface area of the sphere $x^2 + y^2 + z^2 = 16$ between the planes $z = 2$ and $z = 2\sqrt{3}$.



$x = 4\sin\phi\cos\theta$
$y = 4\sin\phi\sin\theta$
$z = 4\cos\phi$

where $0 \leq \theta \leq 2\pi$ and $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{3}$

$\cos\phi = \frac{2\sqrt{3}}{4}$



$$r_\phi \times r_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4\cos\phi\cos\theta & 4\cos\phi\sin\theta & -4\sin\phi \\ -4\sin\phi\sin\theta & 4\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$r_\phi \times r_\theta = \left\langle 16\sin^2\phi\cos\theta,\ 16\sin^2\phi\sin\theta,\ 16\sin\phi\cos\phi\cos^2\theta + 16\sin\phi\cos\phi\sin^2\theta \right\rangle$$

$$r_\phi \times r_\theta = \left\langle 16\sin^2\phi\cos\theta,\ 16\sin^2\phi\sin\theta,\ 16\sin\phi\cos\phi \right\rangle$$

$$|r_\phi \times r_\theta| = \sqrt{16^2\sin^4\phi\cos^2\theta + 16^2\sin^4\phi\sin^2\theta + 16^2\sin^2\phi\cos^2\phi}$$

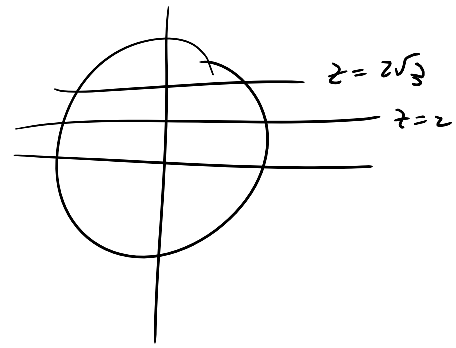
$$|r_\phi \times r_\theta| = \sqrt{16^2\sin^4\phi + 16^2\sin^2\phi\cos^2\phi}$$

$$|r_\phi \times r_\theta| = \sqrt{16^2\sin^2\phi(\sin^2\phi + \cos^2\phi)} = \sqrt{16^2\sin^2\phi} = 16\sin\phi$$

Note: $\sin\phi > 0$ on the given interval of ϕ.

$$S = \int_{\theta=0}^{2\pi}\int_{\phi=\pi/6}^{\pi/3} |r_\phi \times r_\theta|\, d\phi\, d\theta = \int_{\theta=0}^{2\pi}\int_{\phi=\pi/6}^{\pi/3} 16\sin\phi \ \ d\phi\, d\theta = \cdots = 16\pi(\sqrt{3} - 1)$$

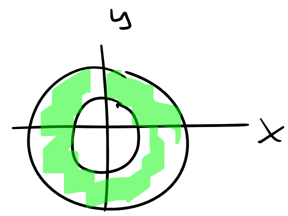
Example: Find the surface area of the sphere $x^2 + y^2 + z^2 = 16$ between the planes $z = 2$ and $z = 2\sqrt{3}$.



$$r(x, y) = \left\langle x, y, \sqrt{16 - x^2 - y^2} \right\rangle$$

if $z = 2$ then this gives $x^2 + y^2 + 4 = 16$ or $x^2 + y^2 = 12$. A circle of radius $2\sqrt{3}$.

if $z = 2\sqrt{3}$ then this gives $x^2 + y^2 + 12 = 16$ or $x^2 + y^2 = 4$. A circle of radius 2.



$-f_x$ $\quad -f_y \quad$ 1

$$r_x \times r_y = \left\langle \frac{x}{\sqrt{16 - x^2 - y^2}}, \frac{y}{\sqrt{16 - x^2 - y^2}}, 1 \right\rangle$$

$$|r_x \times r_y| = \sqrt{\frac{x^2}{16 - x^2 - y^2} + \frac{y^2}{16 - x^2 - y^2} + 1}$$

$$|r_x \times r_y| = \sqrt{\frac{x^2}{16 - x^2 - y^2} + \frac{y^2}{16 - x^2 - y^2} + \frac{16 - x^2 - y^2}{16 - x^2 - y^2}}$$

$$|r_x \times r_y| = \sqrt{\frac{16}{16 - x^2 - y^2}} = \frac{4}{\sqrt{16 - x^2 - y^2}}$$

$$S = \iint_D |r_x \times r_y| \, dA = \iint_D \frac{4}{\sqrt{16 - x^2 - y^2}} dA = \int_{\theta=0}^{2\pi} \int_{r=2}^{2\sqrt{3}} \frac{4r}{\sqrt{16 - r^2}} \, dr d\theta = \cdots = 16\pi(\sqrt{3} - 1)$$