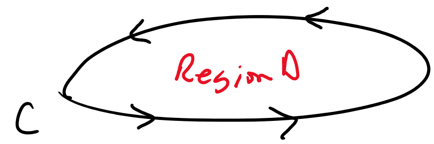
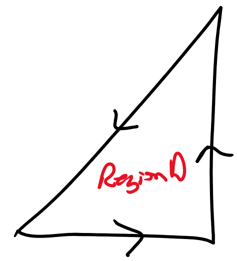
Section 16.4: Green's Theorem

The **positive orientation** of a simple closed curve C refers to a single counterclockwise traversal of C.



Green's Theorem: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then



$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

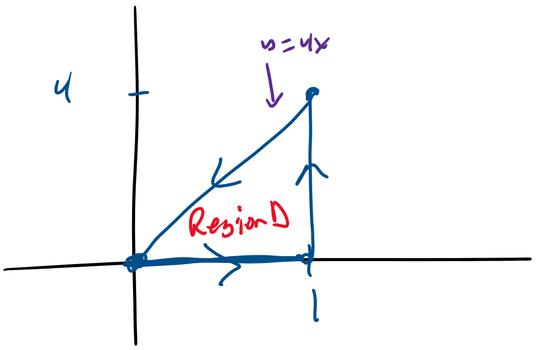
Alternate notations: When $F = \langle P, Q \rangle$ and curve given by $r(t)$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy = \oint_C P dx + Q dy$$

$$\int_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

$P\,dx + Q\,dy \qquad F = \langle P, Q \rangle$

Example: Evaluate $\oint_C x^2y\,dx + x\,dy$ where C is the triangular path from $(0,0)$ to $(1,0)$ to $(1,4)$ to (0.0).



closed path ✓

positive orientation ✓

vector field ✓

Can use Green's thrm

$P = x^2y$

$Q = x$

$P_y = x^2 \qquad Q_x = 1$

$0 \le x \le 1$

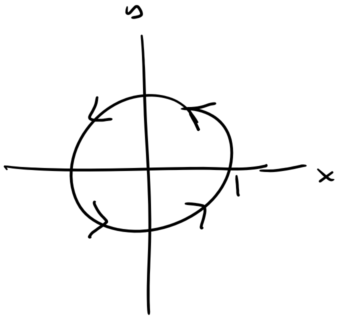
$0 \le y \le 4x$

$$\int_C x^2y\,dx + x\,dy = \iint_D Q_x - P_y\,dA = \iint_D 1 - x^2\,dA$$

$$= \int_{x=0}^{1} \int_{y=0}^{4x} 1 - x^2\,dy\,dx = \dots = 1$$

$$\int F\cdot dr = \int_{*}^{*} F\cdot r'(\theta)\, d\theta$$

Example: Suppose a particle travels one revolution counter-clockwise around the unit circle under the force field $\mathbf{F}(x,y) = \langle \underbrace{e^x - y^3}_{P}, \underbrace{\cos(y) + x^3}_{Q} \rangle$. Find the work done by the field.



$$\text{Work} = \int_C F\cdot dr$$

Closed path ✓
positive orientation ✓
Vector field ✓
} green's Thrm

$F = \langle P, Q \rangle$

$P_y = -3y^2$

$Q_x = 3x^2$

$Q_x - P_y = 3x^2 - -3y^2$

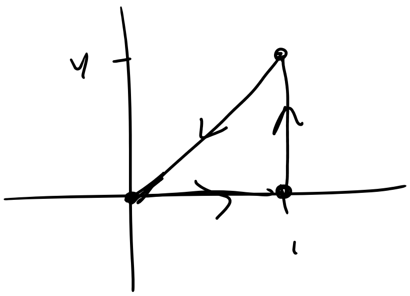
$= 3x^2 + 3y^2$

$$W = \int_C F\cdot dr = \iint_D 3x^2 + 3y^2 \, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 3r^2 \cdot r \, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 3r^3 \, dr\, d\theta = \cdots = \frac{3\pi}{2}$$

Example: Evaluate $\oint_C \overbrace{\left(x^2y+\frac{1}{2}y^2+e^{\sin(x)}\right)}^{P} dx + \overbrace{\left(xy+\frac{1}{3}x^3+x-\arctan(y)\right)}^{Q} dy$
where C is the triangular path from $(0,0)$ to $(1,0)$ to $(1,4)$ to (0.0).

$F = \langle P, Q \rangle$



Closed path ✓

positive orientation ✓

vector field. ✓

$Q_x = y + x^2 + 1$

$P_y = x^2 + y$

$Q_x - P_y = y + x^2 + 1 - (x^2 + y)$

$= 1$

$$\oint_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA = \iint_D 1 \, dA = \frac{1}{2}(1)(4) = \underline{2}$$

Area Using Line Integrals:

$Q_x - P_y = 1$

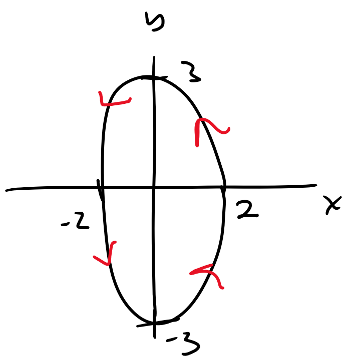
Since $\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$, we need $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

Method 1:
$P = 0$ and $Q = x$
$F = \langle 0, x \rangle$

Method 2:
$P = -y$ and $Q = 0$
$F = \langle -y, 0 \rangle$

Method 3:
$P = \frac{-1}{2} y$ and $Q = \frac{1}{2} x$
$F = \langle -\frac{1}{2} y, \frac{1}{2} x \rangle$

Example: Find the area enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$



$x = 2\cos\theta$

$y = 3\sin\theta$

closed ✓
positive orientation ✓
vector field ✓

choose $F = \langle -\frac{1}{2} y, \frac{1}{2} x \rangle$

$F = \langle -\frac{1}{2} 3\sin\theta, \frac{1}{2} 2\cos\theta \rangle$

$F = \langle -\frac{3}{2}\sin\theta, \cos\theta \rangle$

$r = \langle 2\cos\theta, 3\sin\theta \rangle$

$r' = \langle -2\sin\theta, 3\cos\theta \rangle$

$$\text{Area} = \iint_D 1\, dA = \iint_D Q_x - P_y \, dA = \int_C F \cdot dr = \int_{\theta=0}^{2\pi} 3\sin^2\theta + 3\cos^2\theta \, d\theta$$

$$= \int_{\theta=0}^{2\pi} 3 \, d\theta = 3\theta \Big|_0^{2\pi} = 3(2\pi) - 0 = 6\pi$$

$r(t) = \langle x(t), y(t) \rangle$

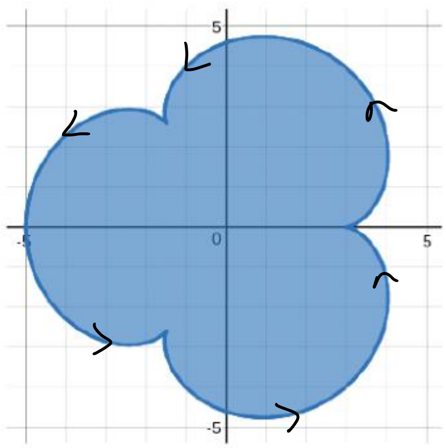
$r' = \langle x', y' \rangle$

Example: Setup the integral(s) that would give the area of the shaded region shown below. The figure is created with the parametric equations:

$F = \langle 0, x \rangle$

$x = 4\cos(t) - \cos(4t), \quad y = 4\sin(t) - \sin(4t)$

$F = 0 + x\,y'$



positive orientation ✓

closed path ✓

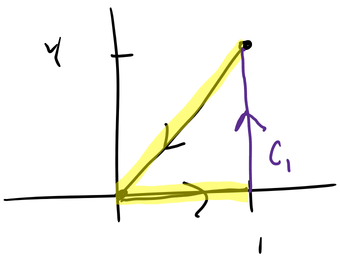
$$\text{Area} = \iint_D 1\, dA = \int_C F \cdot dr$$

$$= \int_0^{2\pi} \left(4\cos(t) - \cos(4t)\right) \cdot \left(4\cos(t) - 4\cos(4t)\right) \, dt$$

Example: Evaluate $\int_C x^2 y\,dx + x\,dy$ where C is the path from (1, 4) to (0, 0) to (1, 0)

P Q

$F = \langle x^2 y, x \rangle$



not a closed path

positive orientation ✓

vector field ✓

Can close the path with C_1 (positive orientation for the Region)

Can now use Green's Thm

and then subtract out $\int_{C_1} F \cdot dr$ to get the final Answer.

$P = x^2 y \qquad Q = x$

$P_y = x^2 \qquad Q_x = 1$

Region D

$0 \le x \le 1$

$0 \le y \le 4x$

$Q_x - P_y = 1 - x^2$

green's Thm

$$\int F \cdot dr = \iint_D 1 - x^2\, dA = \int_{x=0}^{1} \int_{y=0}^{4x} 1 - x^2\, dy\,dx = \cdots = 1$$

added boundary

from (1,0) to (1,4)

need $r(t) = (1-t)\langle 1, 0\rangle + t\langle 1, 4\rangle$

$= \langle 1-t, 0\rangle + \langle t, 4t\rangle$

$r(t) = \langle 1, 4t\rangle$

$r'(t) = \langle 0, 4\rangle$

$F = \langle x^2 y, x\rangle = \langle (1)^2(4t), 1\rangle$

$= \langle 4t, 1\rangle$

$$\int_C F \cdot dr$$

$$= \int_0^1 4t(0) + 1(4)\, dt$$

$$= \int_0^1 4\,dt = 4t \Big|_0^1 = 4$$

Answer

green's thm additional

$$1 - 4 = -3$$