Section 16.3: The Fundamental Theorem for Line Integrals

Recall the Fundamental Theorem of Calculus: $\int_a^b F'(x)dx = F(b) - F(a)$

Theorem: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} =$$

$$F = \nabla f = \langle f_x, f_y, f_z \rangle$$

$$r'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

$$= \int_a^b F \cdot r'(t)\, dt$$

$$= \int_a^b \underbrace{f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}}\, dt$$

This is the chain Rule of $\frac{d}{dt} f(r(t))$

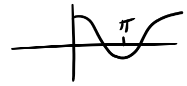
$$= \int_a^b \frac{d}{dt} f(r(t))\, dt$$

$$= f(r(t)) \Big|_a^b = f(\underbrace{r(b)}_{\text{end point of the curve}}) - f(\underbrace{r(a)}_{\text{start point of the curve.}})$$

gradient vector field

Note: The line integral of a conservative vector field (∇f with potential function f) can be evaluated by knowing the endpoints of the curve.

Note: This can also be used on curves that are that are piecewise smooth.



Example: Let $f(x, y) = 3x + x^2y - y^3$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \nabla f$ and C is the curve given by $\mathbf{r}(t) = \langle e^t \sin(t), e^t \cos(t)\rangle$, $0 \le t \le \pi$.

Can use FTLI

$$r(0) = \langle e^0 \sin(0), e^0 \cos(0) \rangle = \langle 0, 1 \rangle \Longrightarrow \text{start point } (0,1)$$

$$r(\pi) = \langle e^\pi \sin(\pi), e^\pi \cos(\pi) \rangle = \langle 0, -e^\pi \rangle \Longrightarrow \text{end point } (0, -e^\pi)$$

$$\int_C F \cdot dr = f(r(\pi)) - f(r(0)) = f(0, -e^\pi) - f(0,1)$$

$$= 0 + 0 - (-e^\pi)^3 - [0 + 0 - (1)^3]$$

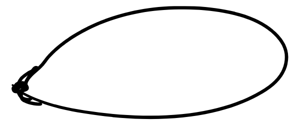
$$= \underline{e^{3\pi} + 1}$$

Pg 3: definitions

Definition: If $\mathbf{F}$ is a continuous vector field with domain D, we say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if and only if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 with the same starting and ending points.

Note: **Line integrals of conservative vectors fields are independent of path.**

Theorem: $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D. (A closed path is one that starts and stops at the same point.)



An interpretation is that the work done by a conservative vector field as an object moves around a closed path is 0.

Question: How do we determine if a vector field is conservative and if so, can we find the potential function?

Theorem: IF $\mathbf{F}(x, y) = \langle P(x,y), Q(x,y)\rangle$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then we have $P_y = Q_x$.

$$F = \nabla f = \langle f_x, f_y \rangle = \langle P, Q \rangle$$

$$f_x = P \qquad P_y = f_{xy} = f_{yx} = Q_x$$

$$f_y = Q \qquad P_y = Q_x$$

Example: Is $\mathbf{F} = \langle \underbrace{3x^2 - 4y}_{P}, \underbrace{4y^2 - 2x}_{Q} \rangle$ a conservative vector field?

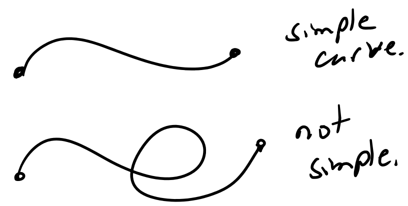
$P = 3x^2 - 4y$ $\qquad P_y = -4$

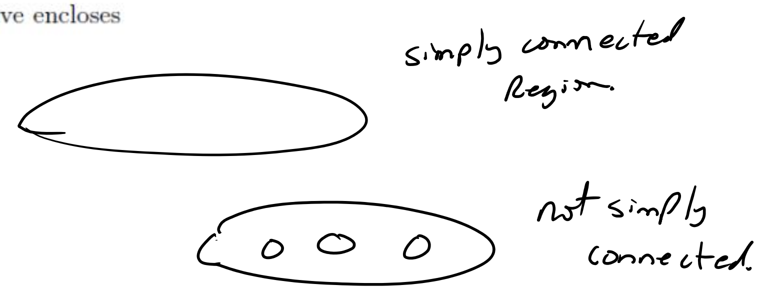
$Q = 4y^2 - 2x$ $\qquad Q_x = -2$

are these equal? NO.

F is <u>not</u> conservative

Definition: A **simple curve** is a curve that does not intersect itself anywhere between its endpoints. A **simply-connected region** in the plane is one that is connected and does not have holes. i.e. every simple closed curve encloses only points in the region.





Theorem: Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field on an **open simply-connected region** D. Suppose that P and Q have continuous first-order derivatives and $P_y = Q_x$ throughout D. Then $\mathbf{F}$ is conservative.

Note: The above criteria to determine if a vector field is conservative works only for $\Re^2$. The criteria for a vector field in $\Re^3$ is found in section 16.5.

Example: Determine whether $\mathbf{F} = \langle \overbrace{x + y^2}^{P}, \overbrace{2xy + y^2}^{Q} \rangle$ is conservative or not.
If so, find a potential function.

$P = x + y^2$ $\quad$ $P_y = 2y$

$Q = 2xy + y^2$ $\quad$ $Q_x = 2y$

equal So on the xy plane F is conservative

$$f_x = P = x + y^2 \longrightarrow f(x,y) = \frac{x^2}{2} + xy^2 + C(y)$$

$$f_y = Q = 2xy + y^2 \longrightarrow f(x,y) = xy^2 + \frac{y^3}{3} + C(x)$$

$$f(x,y) = xy^2 + \frac{x^2}{2} + \frac{y^3}{3}$$

Example: Given that $\mathbf{F} = \langle 4xe^z, \cos(y), 2x^2e^z \rangle$ is conservative. Compute

$P \quad Q \quad R$

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$, for $0 \le t \le \frac{\pi}{2}$.

$$r(0) = \langle \sin(0), 0, \cos(0) \rangle = \langle 0, 0, 1 \rangle$$

$$r\left(\frac{\pi}{2}\right) = \left\langle \sin\left(\frac{\pi}{2}\right), \frac{\pi}{2}, \cos\left(\frac{\pi}{2}\right) \right\rangle = \left\langle 1, \frac{\pi}{2}, 0 \right\rangle$$

$$f_x = P = 4xe^z \longrightarrow f(x,y,z) = 2x^2e^z + c(y,z)$$

$$f_y = Q = \cos(y) \longrightarrow f(x,y,z) = \sin(y) + c(x,z)$$

$$f_z = R = 2x^2e^z \longrightarrow f(x,y,z) = 2x^2e^z + c(x,y)$$

$$f(x,y,z) = 2x^2e^z + \sin(y)$$

$$\int_C F \cdot dr = f\left(r\left(\frac{\pi}{2}\right)\right) - f(r(0))$$

$$= f\left(1, \frac{\pi}{2}, 0\right) - f(0,0,1)$$

$$= 2(1)e^0 + \sin\left(\frac{\pi}{2}\right) - \left[0 + \sin(0)\right]$$

$$= 2 + 1$$

$$= 3$$

$P \quad Q \quad R \qquad F = \nabla f$

Example: Given $\mathbf{F} = \langle 2xy^3 + z^2, 3x^2y^2 + 2yz, y^2 + 2xz \rangle$ is conservative. Find a potential function F.

$P = f_x = 2xy^3 + z^2 \longrightarrow f(x,y,z) = x^2y^3 + xz^2 + c(y,z)$

$Q = f_y = 3x^2y^2 + 2yz \longrightarrow f(x,y,z) = x^2y^3 + y^2z + c(x,z)$

$R = f_z = y^2 + 2xz \longrightarrow f(x,y,z) = zy^2 + xz^2 + c(x,y)$

$f(x,y,z) = x^2y^3 + xz^2 + y^2z$