Section 16.2: Line Integrals

Reminder: In section 13.3 we discussed arc length of a space curve, $\mathbf{r}(t)$, on the interval $a \le t \le b$. The length of the curve, L is given by

$$L = \int_a^b ds = \int_a^b |\mathbf{r}'(t)| \; dt.$$

Line integrals on a plane:

Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ or by the vector function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$.

$$ds = \sqrt{(x')^2 + (y')^2} \; dt$$

$$ds = |r'(t)| \, dt$$

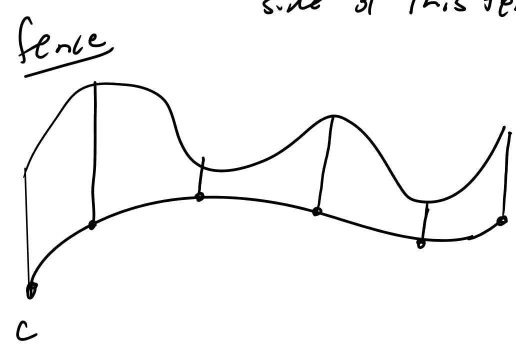
Definition: If f is defined on a smooth curve C, as defined above, then the **line integral of f along C** is

$$\int_C f(x,y)ds = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \qquad \text{if the limit exists.}$$

If f is a continuous function, then we can compute this line integral by

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

Area of one side of this fence



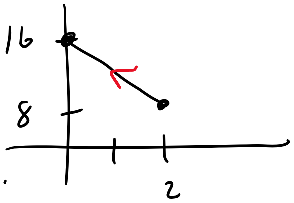
Note: This is sometimes refers to as the **line integral with respect to arc length**.

Note: The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

When $f(x,y) \ge 0$, the line integral of f along C represents the area of one side of the "fence" or "curtain" whose base is C and whose height at any point on the curve is $f(x,y)$. If $f(x,y) = 1$, then the line integral of f along C is the arc length of the curve C.

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{(x')^2 + (y')^2}\,dt$$

Example: Evaluate $\int_C (x^2 + y)\, ds$ where C consists of the line segment from the point $(2, 8)$ to $(0, 16)$



$$m = \frac{16-8}{0-2} = \frac{8}{-2} = -4$$

$$y - 16 = -4(x - 0)$$

$$y = -4x + 16$$

Parametrize the path

$x = t$ $\qquad y = -4t + 16$

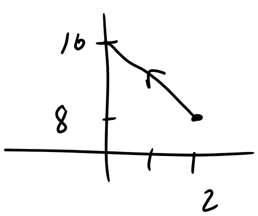
$x' = 1$ $\qquad y' = -4$

$$0 \le t \le 2$$

$$\int_C (x^2 + y)\, ds = \int_0^2 \left[(t)^2 + (-4t + 16)\right] \cdot \sqrt{(1)^2 + (-4)^2}\, dt$$

$$= \int_0^2 (t^2 - 4t + 16)\sqrt{17}\, dt = \cdots = \frac{80\sqrt{17}}{3}$$

Method 2



$$y = -4x + 16$$

$$4x = 16 - y$$

$$x = \frac{16}{4} - \frac{y}{4}$$

$$x = 4 - \frac{1}{4}y$$

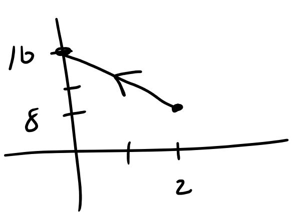
$x = 4 - \frac{1}{4}t$ $\qquad y = t$

$x' = -\frac{1}{4}$ $\qquad y' = 1$

$$8 \le t \le 16$$

$$\int_C x^2 + y\, ds = \int_8^{16} \left[\left(4 - \tfrac{1}{4}t\right)^2 + t\right] \cdot \sqrt{\left(\tfrac{-1}{4}\right)^2 + (1)^2}\, dt = \cdots = \frac{80\sqrt{17}}{3}$$

Method 3



Line segment shortcut.

$$r(t) = (1-t)\langle \text{start point} \rangle + t\langle \text{end point} \rangle$$

$$= (1-t)\langle 2, 8 \rangle + t\langle 0, 16 \rangle$$

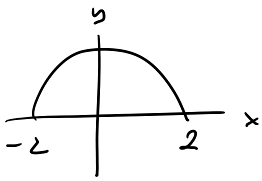
$$= \langle 2-2t,\ 8-8t \rangle + \langle 0,\ 16t \rangle$$

$$r(t) = \langle \underbrace{2-2t}_{x(t)},\ \underbrace{8+8t}_{y(t)} \rangle \qquad 0 \le t \le 1$$

$$r'(t) = \langle -2,\ 8 \rangle$$

$$\int_C (x^2+y)\,ds = \int_0^1 \left[(2-2t)^2 + 8+8t\right] \cdot \sqrt{(-2)^2+(8)^2}\ dt = \cdots = \frac{80\sqrt{17}}{3}$$

Example: Evaluate $\int_C (2+x^2y)ds$, where C is the upper half of the circle $x^2+y^2=4$.



$$x = t \qquad y = \sqrt{4-t^2} \qquad -2 \le t \le 2$$

$$x' = 1 \qquad y' = \frac{1}{2}(4-t^2)^{-1/2} \cdot (-2t)$$

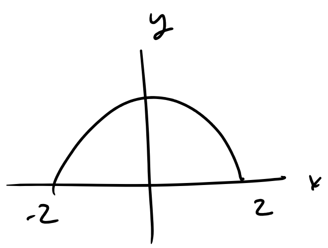
$$y' = \frac{-t}{\sqrt{4-t^2}}$$

$$\int_C (2+x^2y)\,ds = \int_{-2}^{2} \left[2 + (t)^2\sqrt{4-t^2}\right] \cdot \sqrt{(1)^2 + \left(\frac{-t}{\sqrt{4-t^2}}\right)^2}\, dt$$

$$= \int_{-2}^{2} \left(2 + t^2\sqrt{4-t^2}\right)\sqrt{1 + \frac{t^2}{4-t^2}}\, dt$$

$$= \int_{-2}^{2} \left(2+t^2\sqrt{4-t^2}\right)\sqrt{\frac{4}{4-t^2}}\, dt = \int_{-2}^{2} \left(2 + t^2\sqrt{4-t^2}\right)\frac{2}{\sqrt{4-t^2}}\, dt$$

$$= \int_{-2}^{2} \frac{4}{\sqrt{4-t^2}} + 2t^2\, dt$$



$$x = r\cos\theta \qquad y = r\sin\theta$$

$$x = 2\cos\theta \qquad y = 2\sin\theta \qquad 0 \le \theta \le \pi$$

$$x' = -2\sin\theta \qquad y' = 2\cos\theta$$

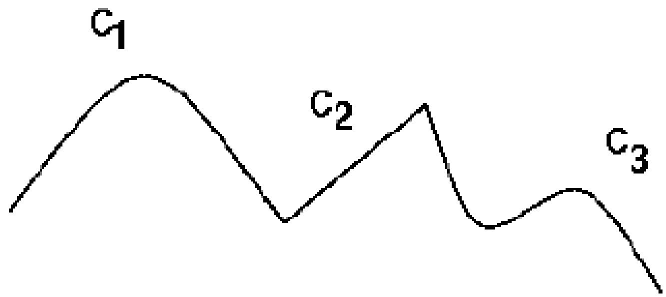
$$\int_C (2+x^2y)\,ds = \int_0^{\pi} \left[2 + (2\cos\theta)^2 \cdot 2\sin\theta\right] \cdot \sqrt{(-2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta$$

$$= \int^{\pi} \left(2 + 8\cos^2\theta\,\sin\theta\right)\sqrt{4\sin^2\theta + 4\cos^2\theta}\, d\theta$$

$$= \int_0^{\pi} \left(2 + 8\cos^2\theta \sin\theta\right) \sqrt{4\sin^2\theta + 4\cos^2\theta}\, d\theta$$

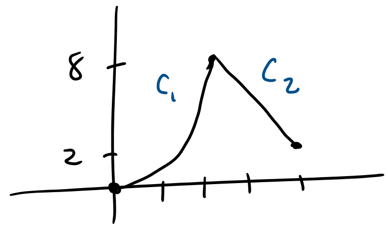
$$= \int_0^{\pi} \left(2 + 8\cos^2\theta \sin\theta\right) \cdot \sqrt{4}\, d\theta = \cdots = 4\pi + \frac{32}{3}$$

Definition: If C is a **piecewise-smooth curve**, that is C is made up a collection of smooth curves where one curve ends then the next curve begins, then line integral of f along C is defined to be the sum of the integrals of f along each smooth piece of C.



$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + \int_{C_3} f(x,y)ds$$

Example: Evaluate $\int_C 2y\,ds$, where C consists of C_1 of $y = x^3$ from $(0,0)$ to $(2,8)$ followed by the line segment from $(2,8)$ to $(4,2)$.



C_1: $x = t \quad y = t^3 \quad 0 \le t \le 2$

$x' = 1 \quad y' = 3t^2$

$$\int_{C_1} 2y\,ds = \int_0^2 2(t^3)\sqrt{(1)^2+(3t^2)^2}\,dt = \int_0^2 2t^3\sqrt{1+9t^4}\,dt = \cdots = \frac{145\sqrt{145}-1}{27}$$

$u = 1 + 9t^4$

C_2: $r(t) = (1-t)\langle 2, 8\rangle + t\langle 4, 2\rangle$

$$= \langle \underbrace{2+2t}_{x(t)}, \underbrace{8-6t}_{y(t)} \rangle \qquad 0 \le t \le 1$$

$r'(t) = \langle 2, -6\rangle$

$$\int_{C_2} 2y\,ds = \int_0^1 2(8-6t)\cdot\sqrt{2^2+(-6)^2}\,dt = \cdots = 20\sqrt{10}$$

Answer: $$\int_C 2y\,ds = \frac{145\sqrt{145}-1}{27} + 20\sqrt{10}$$

Pg 6: line integrals with respect to x and to y

Definition: Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ for $a \le t \le b$.

The **line integral of f along C with respect to x** is $\int_C f(x,y)dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$

The **line integral of f along C with respect to y** is $\int_C f(x,y)dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$

Note: **For these integrals, the orientation of the curve, which direction is is traversed, is important.** If C and $-C$ represent traversing the same curve but in different directions, then

$$\int_{-C} f(x,y)dx = -\int_C f(x,y)dx$$

Pg 7

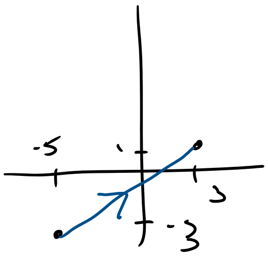
Example: Evaluate $\displaystyle\underbrace{\int_C y^2 dx + x dy}$, where C is the line segment from $(-5,-3)$ to $(3,1)$.

$\hookrightarrow \displaystyle\int_C y^2 dx + \int_C x dy$

$0 \le t \le 1$

$r(t) = (1-t)\langle -5,-3\rangle + t\langle 3,1\rangle$

$r(t) = \langle \underbrace{-5+8t}_{x}, \underbrace{-3+4t}_{y} \rangle$



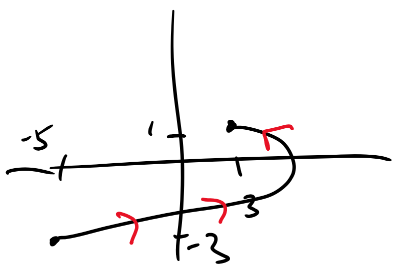
$r'(t) = \langle 8, 4 \rangle$

$x'(t) = 8\,dt \qquad y'(t) = 4\,dt$

$$\int_C y^2 dx + x dy = \int_0^1 (-3+4t)^2 \cdot 8\,dt + (-5+8t)\cdot 4\,dt$$

$$= \int_0^1 (-3+4t)^2 \cdot 8 + (-5+8t)\cdot 4 \quad dt = \cdots = \frac{44}{3}$$

Example: Evaluate $\int_C y^2 dx + x dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.



$x = 4 - t^2 \qquad y = t \qquad -3 \le t \le 1$

$x' = -2t \qquad y' = 1$

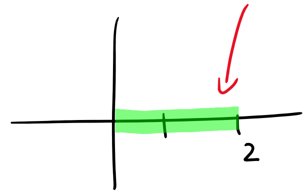
$$\int_C y^2 dx + x dy = \int_{-3}^{1} (t)^2 \cdot (-2t) + (4 - t^2) \cdot 1 \quad dt$$

$$= \int_{-3}^{1} -2t^3 + 4 - t^2 \, dt = \cdots = \frac{140}{3}$$

Example: Consider $f(x, y) = x^2$ and C be the smooth curve $r(t) = \langle t, 0\rangle$ for $0 \le t \le 2$. Then $x(t) = t$ and $y(t) = 0$ and $x'(t) = 1\ dt$

$$\int_C f(x,y)dx = \int_0^2 t^2 * 1\ dt = \int_0^2 t^2 dt$$

Compare this to integrating $y = x^2$ on the interval $[0, 2]$ which is $\int_0^2 x^2 dx$

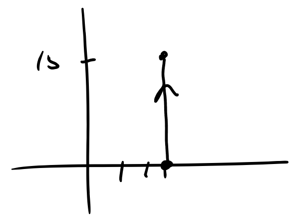


C is part of the x-axis.

$dx = x'(t)\ dt$

Example: The curve C is the line segment from $(3, 0)$ to $(3, 15)$.

Compute $\int_C (x^2 + 2y)dx$



$0 \le t \le 1$

$r(t) = \langle 3, 15t \rangle$

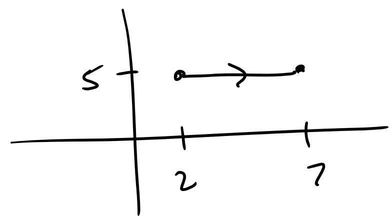
$r' = \langle 0, 15 \rangle$

$dx = 0\ dt$

$$= \int_0^1 \left[(3)^2 + 2(15t)\right] 0\ dt = 0$$

Example: The curve C is the line segment from $(2, 5)$ to $(7, 5)$.

Compute $\int_C (x^2 + 2y)dy = 0$



Pg 10: line integrals in space

Line Integrals in Space:

Let C be a smooth curve defined by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \le t \le b$. The line integral of f along C is

$$\int_C f(x,y,z)ds = \int_a^b f(x(t), y(t), z(t)) \ \ |\mathbf{r}'(t)| \ dt$$

$$|r'(t)| = \sqrt{(x')^2 + (y')^2 + (z')^2}$$

Note: The line integral of f with respect to x, with respect to y, and with respect to z are defined in a manner similar to before.

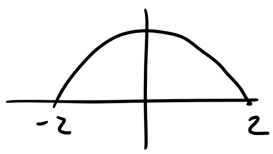
Physical interpretation of a line integral: Let $\rho(x,y)$ represent the linear density at a point (x,y) of a thin wire shaped like the curve C.

Then the mass of the wire is $m = \int_C \rho(x,y)ds$

The center of mass $(\overline{x}, \overline{y})$ is

$$\overline{x} = \frac{1}{m}\int_C x\rho(x,y)ds \text{ and } \overline{y} = \frac{1}{m}\int_C y\rho(x,y)ds$$

Example: A thin wire with linear density $\rho(x,y) = 2 + x^2 y$ takes the shape of the semicircle $x^2 + y^2 = 4$, $y \geq 0$. Find the mass of this wire.

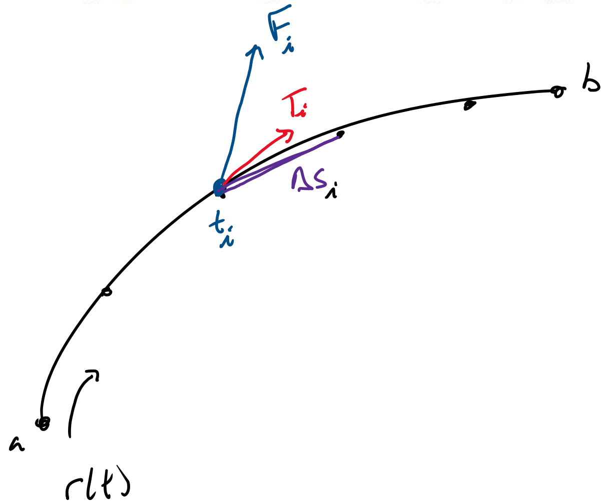


mass $\int_C 2 + x^2 y \, ds = 4\pi + \frac{32}{3}$ see earlier example

Pg 12: line integrals in vector fields

Line Integrals of Vector Fields

Suppose **F** is a continuous vector field (i.e. force field). Find the work done moving a particle along the curve C given by $\mathbf{r}(t)$ for $a \le t \le b$.



$$W = F \cdot d$$

T_i is the unit tangent vector at t_i

distance $d \approx T_i \, \Delta s_i$

$$W_i = F_i \cdot T_i \, \Delta s_i$$

$$W \approx \sum F_i \cdot T_i \, \Delta s_i$$

$$\text{work} = \int_a^b F \cdot T \, ds$$

Definition: Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of F along C** is

$$\int_c \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_c \mathbf{F} \cdot \mathbf{T}\, ds$$

$$T = \frac{r'(t)}{|r'(t)|}$$

$$ds = |r'(t)|\, dt$$

$$F \cdot T ds = F \cdot \frac{r'(t)}{|r'(t)|} |r'(t)| dt$$

Note: If the orientation of the curve is changed, i.e. C is replaced by $-C$, then the unit tangent vector is replaced by its negative. Thus

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

Example: Find the work done by the force field $\mathbf{F}$ in moving a particle along the curve C.

$\mathbf{F}(x, y, z) = \langle xy, yz, xz \rangle$

C: $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$.

$x = t$
$y = t^2$
$z = t^3$

$F(r(t)) = \langle t \cdot t^2, t^2 \cdot t^3, t \cdot t^3 \rangle$
$= \langle t^3, t^5, t^4 \rangle$

$r'(t) = \langle 1, 2t, 3t^2 \rangle$

$$W = \int_C F \cdot d\vec{r} = \int_0^1 F \cdot r'(t)\, dt$$

$$= \int_0^1 t^3 + t^5(2t) + t^4(3t^2)\, dt$$

$$= \int_0^1 t^3 + 2t^6 + 3t^6\, dt = \int_0^1 t^3 + 5t^6\, dt = \cdots = \frac{27}{28}$$

Relationship between a line integral over a vector field and line integrals with respect to x, y, and z.

Let $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ with $\longrightarrow \langle P, Q, R\rangle$

C defined by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$ $\longrightarrow$ $r'(t) = \langle dx, dy, dz\rangle$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle = \underbrace{\int_C P dx + Q dy + R dz}$$