Cylindrical Coordinates:
A Cartesian point (x, y, z) is represented by (r, θ, z) in the Cylindrical Coordinate System. Where (r, θ) represent the polar coordinates for the point (x, y) and z is the distance above or below the $xy-$plane.

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$ $\quad$ $z = z$ $\quad$ $r^2 = x^2 + y^2$ $\quad$ $\tan\theta = \dfrac{y}{x}$

Note: Cylindrical coordinates are useful in problems that involve symmetry about the $z-$axis.

Example: Find the cylindrical coordinates for the point $\overset{x}{(-1}, \overset{y}{\sqrt{3}}, \overset{z}{2)}$

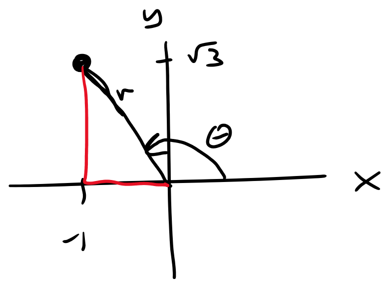
$r = \sqrt{x^2+y^2}$

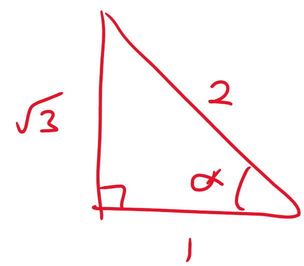
$r = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$= \sqrt{1+3}$

$= \sqrt{4}$

$r = 2$



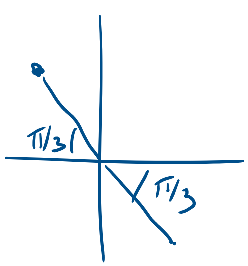


$\theta = \dfrac{2\pi}{3}$ $\qquad$ $\alpha = \dfrac{\pi}{3}$

(r, θ, z)

$\left(2, \dfrac{2\pi}{3}, 2\right)$ $\qquad$ $\left(-2, \dfrac{-\pi}{3}, 2\right)$

$\left(2, \dfrac{2\pi}{3} + 2\pi, 2\right)$



$x^2+y^2=r^2$

Example: Write the equations in cylindrical coordinates.

A) $z = 12 - 4x^2 - 4y^2$ $= 12 - 4(x^2+y^2)$

$z = 12 - 4r^2$

$z = 3x + 2y$

$z = 3r\cos\theta + 2r\sin\theta$

Cone.

B) $z = \sqrt{3x^2 + 3y^2}$

$z = \sqrt{3r^2}$

$z = \sqrt{3}\, r$

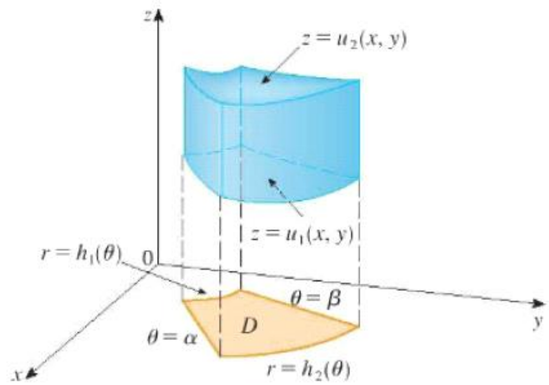
Triple Integrals in Cylindrical Coordinates

Suppose that E is a solid whose image D on the $xy-$plane can be described in polar coordinates.

$E = \{(x, y, z)|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$

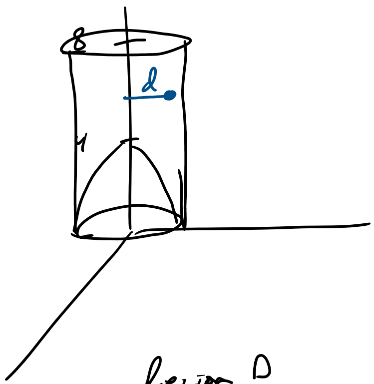
and $D = \{(r, \theta)|a \leq \theta \leq b, h1(\theta) \leq r \leq h2(\theta)\}$

If $f(x, y, z)$ is continuous over the solid E,
$g_1(r, \theta) = u_1(r\cos\theta, r\sin\theta)$, and
$g_2(r, \theta) = u_2(r\cos\theta, r\sin\theta)$ then



$$\iiint_E f(x, y, z)dV = \int_{\theta=a}^{b} \int_{r=h_1(\theta)}^{h_2(\theta)} \int_{z=g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\ r\ dzdrd\theta$$

Example A solid lies with in the cylinder $x^2 + y^2 = 4$, below the plane $z = 8$, and above the paraboloid $z = 4 - x^2 - y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.



Top $z = 8$

Bottom $z = 4 - x^2 - y^2 = 4 - r^2$

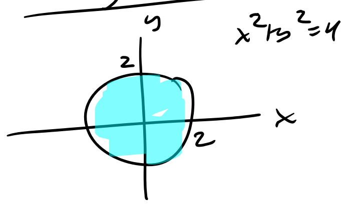
density function

$\sigma(x,y,z) = Kd$

$= K\sqrt{x^2+y^2}$

$$\text{mass} = \iiint_E \sigma(x,y,z)\, dV$$

Region D



$0 \le \theta \le 2\pi$

$0 \le r \le 2$

$$\text{mass} = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=4-r^2}^{8} K\sqrt{r^2} \cdot r \, dz\, dr\, d\theta$$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=4-r^2}^{8} Kr^2 \, dz\, dr\, d\theta$$

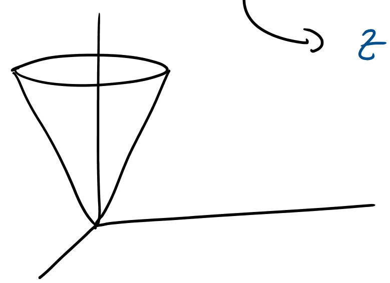
$$= \cdots = 2K\pi\left(\frac{32}{3} + \frac{32}{5}\right)$$

$z^2 = x^2+y^2$ cone.

Example: Evaluate $\displaystyle\underbrace{\int_{x=0}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=\sqrt{x^2+y^2}}^{2}}(x^2+y^2)\,dz\,dy\,dx$

Top $z=2$

Bottom $z=\sqrt{x^2+y^2}$ top part of the cone. $\rightarrow z=\sqrt{r^2}=r$



Region D

$-\underbrace{\sqrt{4-x^2}\le y\le\sqrt{4-x^2}}$ $\quad 0\le x\le 2$

$y=\sqrt{4-x^2}$

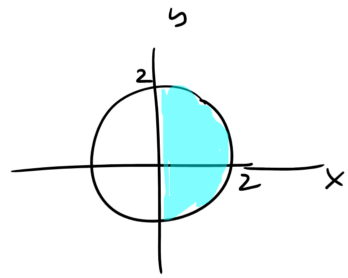
$y^2=4-x^2$

$x^2+y^2=4$

$0\le r\le 2$

$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$\frac{3\pi}{2}\le\theta\le\frac{5\pi}{2}$



Example: Evaluate $\displaystyle\int_{x=0}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$ $\displaystyle=\int_{\theta=-\frac{\pi}{2}}^{\pi/2}\int_{r=0}^{2}\int_{z=r}^{2} r^2\cdot r\,dz\,dr\,d\theta$

$= \ldots = \frac{8\pi}{5}$