Section 15.6: Triple Integrals

Let B be a rectangular box such that $B = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$, i.e. $B = [a, b] \times [c, d] \times [r, s]$.

x y z

Definition: The **triple integral** of $f(x, y, z)$ over the box B is

$$\iiint_B f(x,y,z)dV = \lim_{||P|| \to 0} \sum_i \sum_j \sum_k f(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

if this limit exists.

Note: The volume of solid E is given by $\iiint_E 1 \, dV$.

Fubini's Theorem for Triple Integrals: If f is continuous on the rectangular box

x y z

$B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x,y,z)dV = \int_{z=r}^{s} \int_{y=c}^{d} \int_{x=a}^{b} f(x,y,z) \, dxdydz = \int_{x=a}^{b} \int_{z=r}^{s} \int_{y=c}^{d} f(x,y,z) \, dydzdx$$

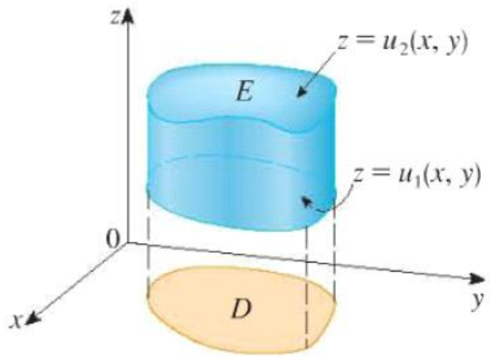
Triple Integral over General Regions:

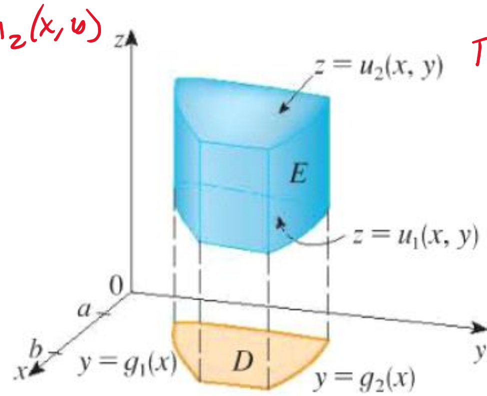
When using a non-rectangular solid, we consider the projection (image) the solid makes on the different coordinate planes.

A **Type I** region is the solid $E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$

Bottom Top

$u_1(x,y) \leq z \leq u_2(x,y)$





Top.

Bottom.

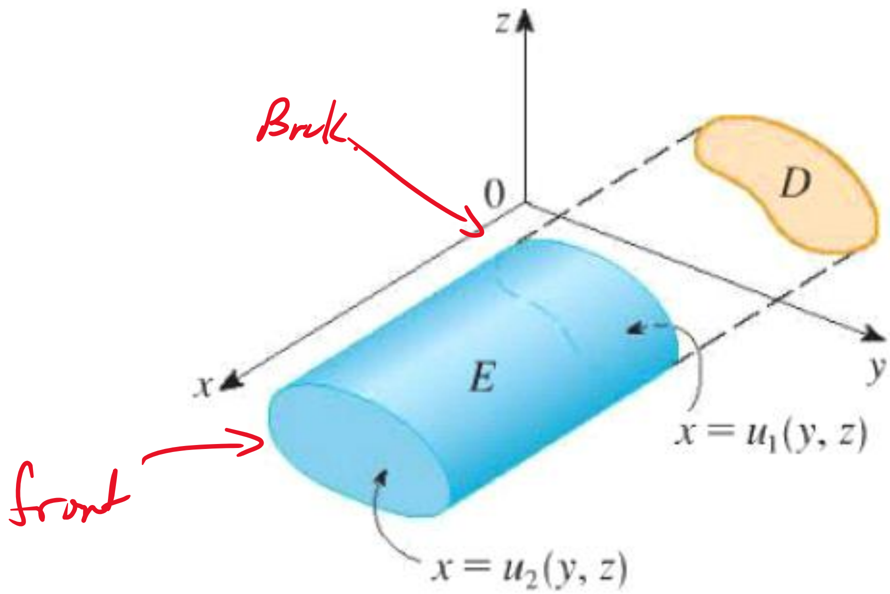
$a \leq x \leq b$

$g_1(x) \leq y \leq g_2(x)$

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz \right] dA$$

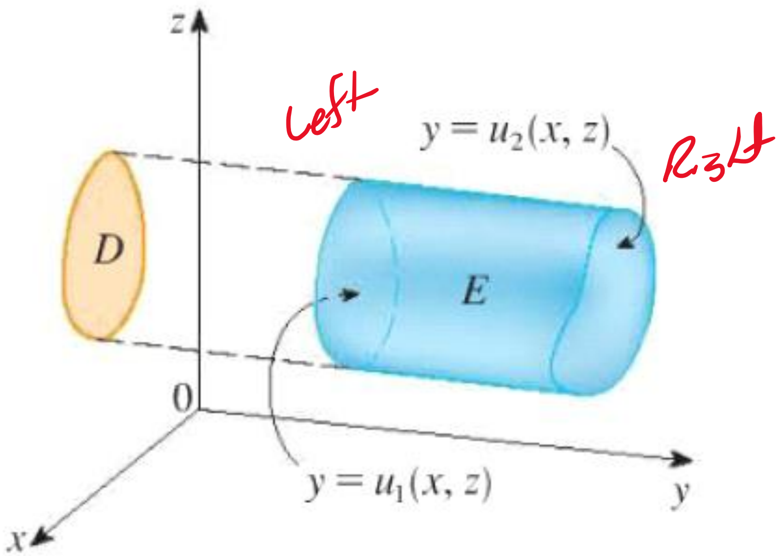
$$= \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} \int_{z=u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

A **Type 2** region is the solid $E = \{(x,y,z) | (y,z) \in D, u_1(y,z) \le x \le u_2(y,z)\}$



$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx \right] dA$$

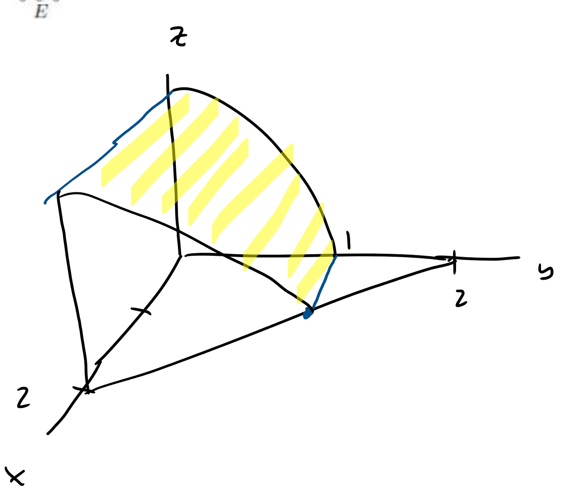
A **Type 3** region is the solid $E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\}$



$$\iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)dy \right] dA$$

Example: Given E is the solid bounded by the plane $x + y = 2$ and the cylinder $y^2 + z^2 = 1$ in the first octant. Setup the triple integral with different projections on the different coordinate planes.

$$\iiint_E z \, dV$$

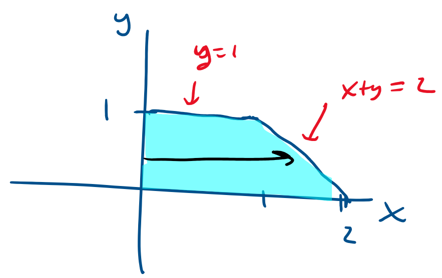


project on xy plane

Top: $z = \sqrt{1-y^2}$

Bottom: $z = 0$

Region D



$dy\,dx \uparrow$ $\qquad$ $dx\,dy \rightarrow$

$0 \le y \le 1$

$0 \le x \le 2-y$

$$\iiint_E z \, dV = \iint_D \left[\int_0^{\sqrt{1-y^2}} z \, dz \right] dA$$

$$= \int_{y=0}^{1} \int_{x=0}^{2-y} \underbrace{\int_{z=0}^{\sqrt{1-y^2}} z \, dz}\, dx\, dy = \int_{y=0}^{1} \int_{x=0}^{2-y} \left. \frac{1}{2} z^2 \right|_{z=0}^{\sqrt{1-y^2}} dx\, dy$$

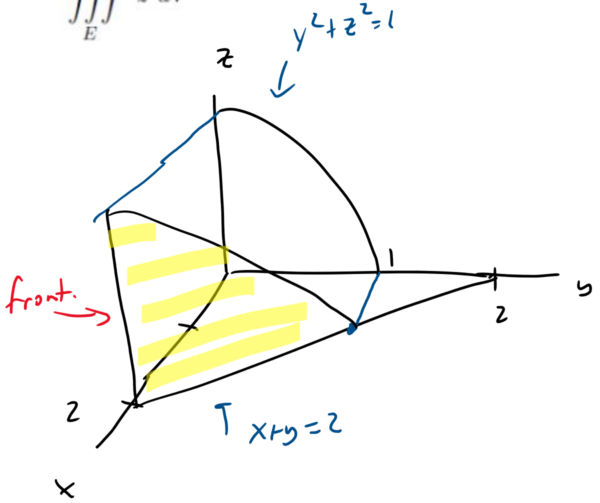
$$= \int_{y=0}^{1} \int_{x=0}^{2-y} \underbrace{\frac{1}{2}}(1-y^2)\, dx\, dy = \frac{1}{2} \int_{y=0}^{1} \left. x(1-y^2) \right|_{x=0}^{2-y} dy$$

$$= \frac{1}{2} \int_{y=0}^{1} (2-y)(1-y^2)\, dy = \frac{1}{2} \int_{y=0}^{1} 2 - 2y^2 - y + y^3 \, dy$$

$$= \frac{1}{2}\left[2y - \frac{2y^3}{3} - \frac{y^2}{2} + \frac{y^4}{4}\right]\Big|_0^1 = \frac{1}{2}\left(2 - \frac{2}{3} - \frac{1}{2} + \frac{1}{4}\right) = \frac{13}{24}$$

Example: Given E is the solid bounded by the plane $x + y = 2$ and the cylinder $y^2 + z^2 = 1$ in the first octant. Setup the triple integral with different projections on the different coordinate planes.

$$\iiint_E z\, dV$$

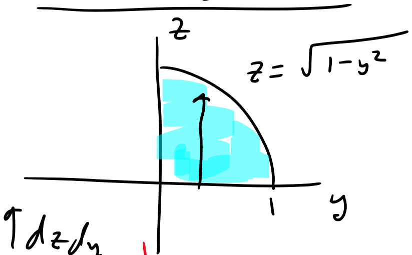


project on yz plane

front: $x = 2-y$

Back: $x = 0$

Region D



$\uparrow dz\,dy$

$0 \le y \le 1$

$0 \le z \le \sqrt{1-y^2}$

Looks polar.

$0 \le \theta \le \frac{\pi}{2}$

$0 \le r \le 1$

$$\boxed{\begin{aligned} y &= r\cos\theta \\ z &= r\sin\theta \\ y^2+z^2 &= r^2 \end{aligned}}$$

$$\iiint_E z\, dV = \iint_D \left[\int_{x=0}^{2-y} z\, dx \right] dA$$

$$= \int_{y=0}^{1} \int_{z=0}^{\sqrt{1-y^2}} \int_{x=0}^{2-y} z\, dx\, dz\, dy$$

$$\iint_D xz \Big|_0^{2-y} dA$$

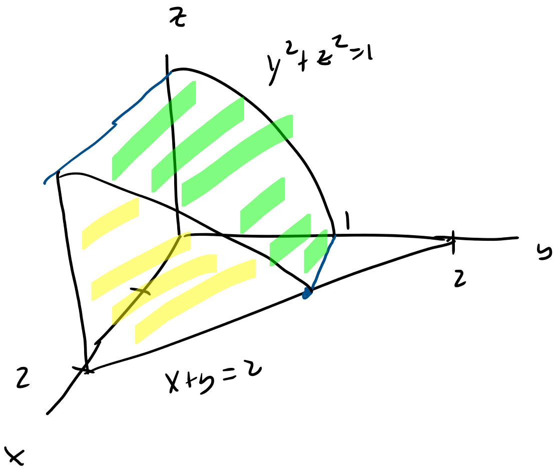
$$= \iint_D (2-y)z\, dA$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} (2 - r\cos\theta)\, r\sin\theta \cdot r\, dr\, d\theta$$

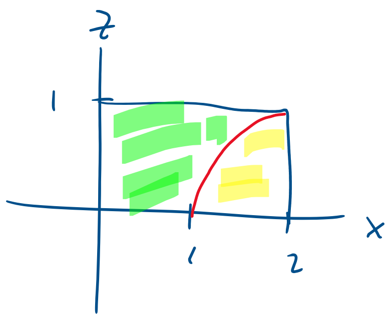
Example: Given E is the solid bounded by the plane $x + y = 2$ and the cylinder $y^2 + z^2 = 1$ in the first octant. Setup the triple integral with different projections on the different coordinate planes.

$$\iiint_E z \, dV$$

project on xz plane



Region D



Left: $y = 0$

Right: $y =$

Two different Right sides

Difficult for this part.

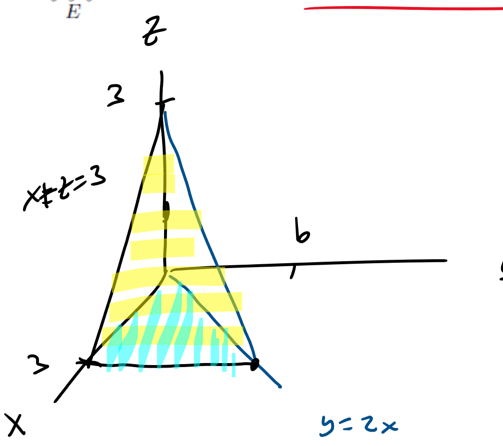
use other method.

Example: Given E is the solid bounded by the planes $z = 3 - x$, $z = 0$, $y = 0$, and $y = 2x$.

Rewrite $\iiint\limits_E f(x, y, z)\, dV$ as 6 different interated integrals.

$z = 3 - x$

$x + z = 3$

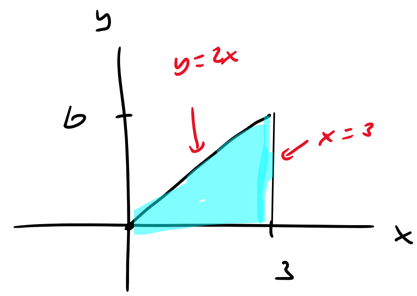


project on xy plane.

top: $z = 3 - x$

Bottom: $z = 0$

Region D



$\uparrow dy\,dx$

$0 \le x \le 3$

$0 \le y \le 2x$

$\overrightarrow{dx\,dy}$

$\frac{y}{2} \le x \le 3$

$0 \le y \le 6$

$$\iiint\limits_E f(x,y,z)\, dV = \int_{x=0}^{3} \int_{y=0}^{2x} \int_{z=0}^{3-x} f(x,y,z)\, dz\, dy\, dx$$

$$= \int_{y=0}^{6} \int_{x=\frac{y}{2}}^{3} \int_{z=0}^{3-x} f(x,y,z)\, dz\, dx\, dy$$

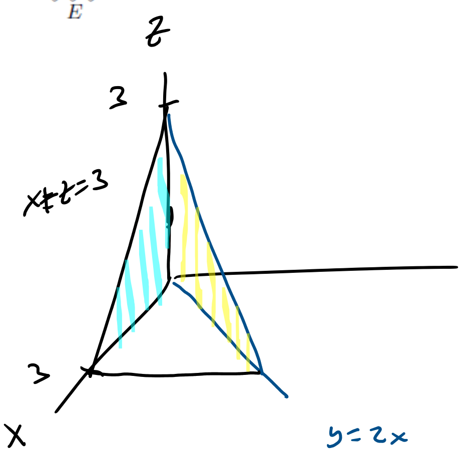
Pg 6 cont

Example: Given E is the solid bounded by the planes $z = 3 - x$, $z = 0$, $y = 0$, and $y = 2x$.

$z = 3 - x$

$x + z = 3$

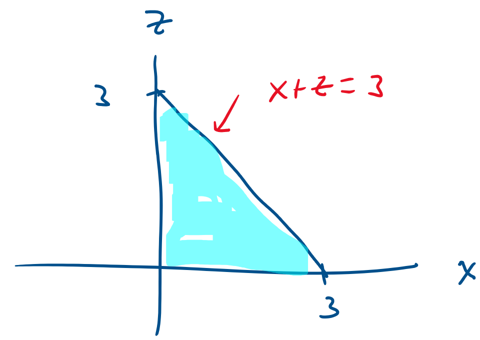
Rewrite $\iiint_E f(x,y,z)\,dV$ as 6 different interated integrals.



project on xz plane

Left $y = 0$

y Right $y = 2x$



$dx\,dz$

$0 \le x \le 3 - z$

$0 \le z \le 3$

$dz\,dx$

$0 \le x \le 3$

$0 \le z \le 3 - x$

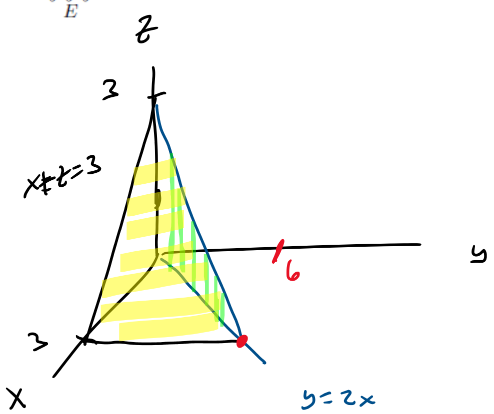
$$\int_{z=0}^{3} \int_{x=0}^{3-z} \int_{y=0}^{2x} f(x,y,z)\,dy\,dx\,dz$$

$$\int_{x=0}^{3} \int_{z=0}^{3-x} \int_{y=0}^{2x} f(x,y,z)\,dy\,dz\,dx$$

Example: Given E is the solid bounded by the planes $z = 3 - x$, $z = 0$, $y = 0$, and $y = 2x$.

Rewrite $\iiint_E f(x, y, z)\, dV$ as 6 different interated integrals.

$z = 3 - x$
$x + z = 3$

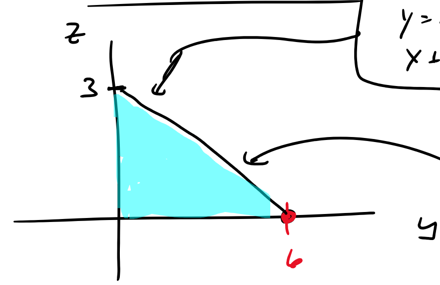


project on yz plane

Front $x = 3 - z$

Back $x = \frac{y}{2}$

Region D



Straight Line since Intersection of 2 planes $y = 2x$ and $x + z = 3$

$y - y_1 = m(x - x_1)$

need formula

$dz\,dy$

$0 \le y \le 6$

$0 \le z \le -\frac{1}{2}y + 3$

$dy\,dz$

$0 \le z \le 3$

$0 \le y \le 6 - 2z$

y z
(0, 3)
(6, 0)

$m = \frac{3-0}{0-6}$

$m = -\frac{1}{2}$

$z - z_1 = m(y - y_1)$

$z - 3 = -\frac{1}{2}(y - 0)$

$z = -\frac{1}{2}y + 3$

$2z = -y + 6$

$y = 6 - 2z$

method 2

$x + z = 3$

$y = 2x \to x = \frac{1}{2}y$

now sub.

$\frac{1}{2}y + z = 3$

$z = 3 - \frac{1}{2}y$

$y + 2z = 6$

$y = 6 - 2z$

$$\int_{y=0}^{6} \int_{z=0}^{-\frac{1}{2}y+3} \int_{x=\frac{y}{2}}^{3-z} f(x,y,z)\, dx\, dz\, dy$$

$$\int_{z=0}^{3} \int_{y=0}^{6-2z} \int_{x=\frac{y}{2}}^{3-z} f(x,y,z)\, dx\, dy\, dz$$

Pg 6 results

project on xy plane

$$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{3}\int_{y=0}^{2x}\int_{z=0}^{3-x} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_{y=0}^{6}\int_{x=\frac{y}{2}}^{3}\int_{z=0}^{3-x} f(x,y,z)\,dz\,dx\,dy$$

project on xz plane

$$\int_{z=0}^{3}\int_{x=0}^{3-z}\int_{y=0}^{2x} f(x,y,z)\,dy\,dx\,dz$$

$$\int_{x=0}^{3}\int_{z=0}^{3-x}\int_{y=0}^{2x} f(x,y,z)\,dy\,dz\,dx$$

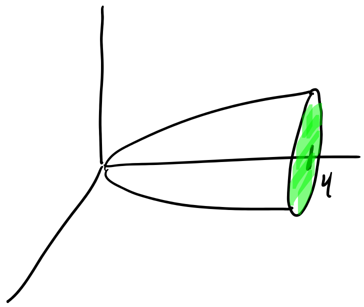
project on yz plane

$$\int_{y=0}^{6}\int_{z=0}^{-\frac{1}{2}y+3}\int_{x=\frac{y}{2}}^{3-z} f(x,y,z)\,dx\,dz\,dy$$

$$\int_{z=0}^{3}\int_{y=0}^{6-2z}\int_{x=\frac{y}{2}}^{3-z} f(x,y,z)\,dx\,dy\,dz$$

Example: Given E is the solid bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. Compute

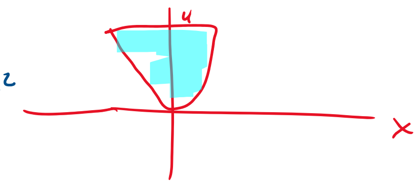
$$\iiint_E \sqrt{x^2+z^2}\, dV$$



project onto xy plane

top $z = \sqrt{y - x^2}$

Bottom $z = -\sqrt{y-x^2}$



$$\iint_D \left[\int_{z=-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz \right] dA$$

not a good choice.

project onto xz plane

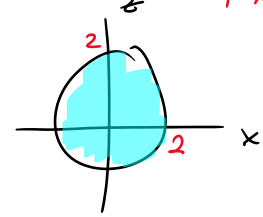
Left $y = x^2+z^2$

Right $y = 4$

$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left[\int_{y=x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy \right] dA$$

Region D

$y = x^2+z^2$



:)

$$\int_{x=-2}^{2} \int_{z=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{y=x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dz\, dx$$

$dz\, dx$

$-\sqrt{4-x^2} \le z \le \sqrt{4-x^2}$

$-2 \le x \le 2$

polar

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$x = r\cos\theta$

$z = r\sin\theta$

$x^2+z^2 = r^2$

$$\iint_D y\sqrt{x^2+z^2}\,\Big|_{y=x^2+z^2}^{4} dA$$

$$= \iint_D 4\sqrt{x^2+z^2} - (x^2+z^2)\sqrt{x^2+z^2}\, dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (4r - r^2\cdot r)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 4r^2 - r^4\, dr\, d\theta = \dots = \frac{128\pi}{15}$$

$$\int_{\theta=0} \int_{r=0} 4r^2 - r^4 \, dr \, d\theta = \cdots \frac{\quad}{15}$$

$$\iiint_E 1 \, dV = \text{volume of solid } E$$

Applications of Triple Integrals:
Given a solid E with density function $\rho(x, y, z)$.

mass: $m = \iiint_E \rho(x, y, z) \, dV$

moments about the yz coordinate plane: $M_{yz} = \iiint_E x \, \rho(x, y, z) \, dV$

moments about the xz coordinate plane: $M_{xz} = \iiint_E y \, \rho(x, y, z) \, dV$

moments about the xy coordinate plane: $M_{xy} = \iiint_E z \, \rho(x, y, z) \, dV$

center of Mass: $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$

6. Rewrite the integral $\int_0^1 \int_0^{2-2y} \int_0^{4-x^2} f(x,y,z)\, dz\,dx\,dy$ in the order of **dy dx dz.**

project on xz plane

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projecting on xy plane

Top $z = 4 - x^2$

Bottom $z = 0$

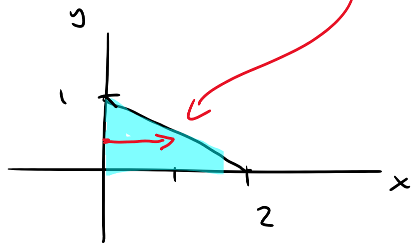
$\overrightarrow{dx\,dy}$

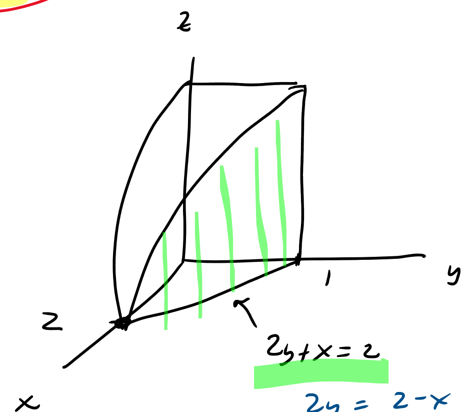
$\underbrace{0 \le}_{x=0} x \underbrace{\le 2-2y}$

$x = 2 - 2y$

$2y + x = 2$

$0 \le y \le 1$





$2y + x = 2$

$2y = 2 - x$

$y = \frac{1}{2}(2 - x)$

Left $y = 0$

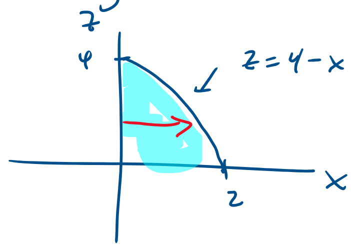
Right $y = 1 - \frac{x}{2}$

$\overrightarrow{dx\,dz}$

$0 \le x \le \sqrt{4-z}$

$0 \le z \le 4$

Region D



$x^2 = 4 - z$

$x = +\sqrt{4-z}$

Answer. $\displaystyle\int_{z=0}^{4} \int_{x=0}^{\sqrt{4-z}} \int_{y=0}^{1-\frac{x}{2}} f(x,y,z)\, dy\, dx\, dz$