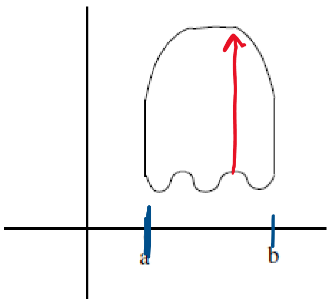
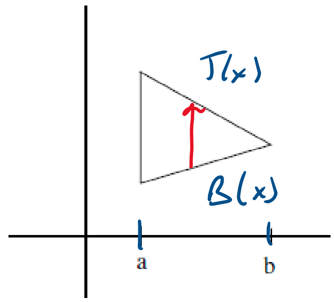
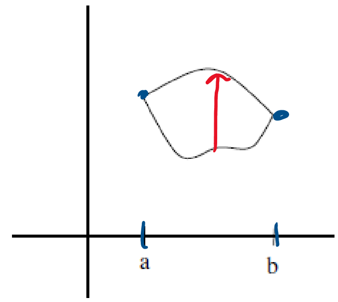
Section 15.2: Double Integrals over General Regions

Definition: A plane region D is said to be of **Type I** if it lies between two continuous functions of x, that is

$D = \{(x, y)|\ a \leq x \leq b,\ B(x) \leq y \leq T(x)\}$







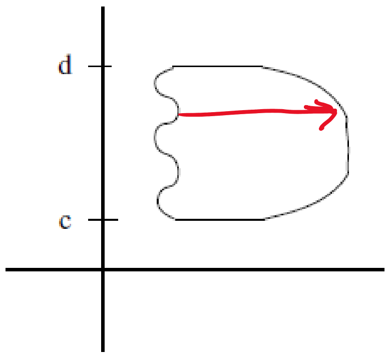
Theorem: If f is continuous on a type I region D such that $D = \{(x, y)|\ a \leq x \leq b, B(x) \leq y \leq T(x)\}$ then

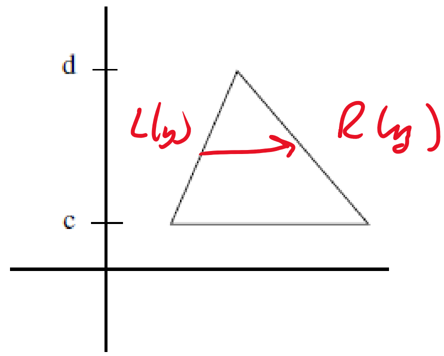
$$\iint_D f(x,y)dA = \int_{x=a}^{b} \int_{y=B(x)}^{T(x)} f(x,y)\ dydx$$

$dy\,dx$ ↑

Definition: A plane region D is said to be of **Type II** if it lies between two continuous functions of y, that is

$$D = \{(x, y)|\ c \leq y \leq d,\ L(y) \leq x \leq R(y)\}$$





Theorem: If f is continuous on a type II region D such that $D = \{(x, y)|\ c \leq y \leq d, L(y) \leq x \leq R(y)\}$then

$$\iint\limits_D f(x, y)dA = \int\limits_{y=c}^{d} \int\limits_{x=L(y)}^{R(y)} f(x, y)\ dxdy$$

$\longrightarrow$

dxdy

Example: If D is the region bounded by $y = 4x$ and $y = x^2$ evaluate

$$\iint_D x + y\, dA$$

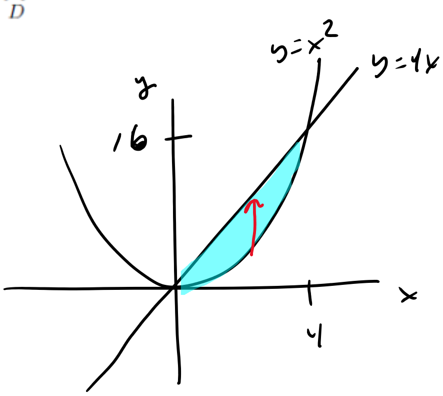
Intersection

$4x = x^2$

$0 = x^2 - 4x$

$0 = x(x-4)$

$x=0 \quad x=4$



dy dx method

$0 \le x \le 4$

$x^2 \le y \le 4x$

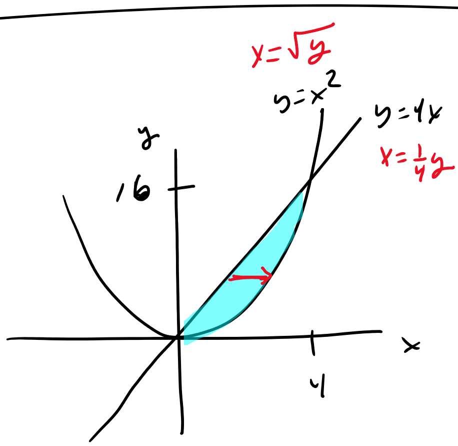
$$\iint_D x+y\, dA = \int_{x=0}^{4} \int_{y=x^2}^{4x} x+y\, dy\, dx = \int_{x=0}^{4} xy + \tfrac{1}{2}y^2 \Big|_{y=x^2}^{4x} dx$$

$$= \int_{x=0}^{4} x(4x) + \tfrac{1}{2}(4x)^2 - \left[x \cdot x^2 + \tfrac{1}{2}(x^2)^2\right] dx$$

$$= \int_{x=0}^{4} 4x^2 + 8x^2 - x^3 - \tfrac{1}{2}x^4\, dx = \int_{x=0}^{4} 12x^2 - x^3 - \tfrac{1}{2}x^4\, dx$$

$$= \left(\frac{12x^3}{3} - \frac{x^4}{4} - \frac{1}{10}x^5\right)\Big|_{x=0}^{4} = 4(4)^3 - \tfrac{1}{4}(4)^4 - \tfrac{1}{10}(4)^5$$

$$= 89.6$$



dx dy

$0 \le y \le 16$

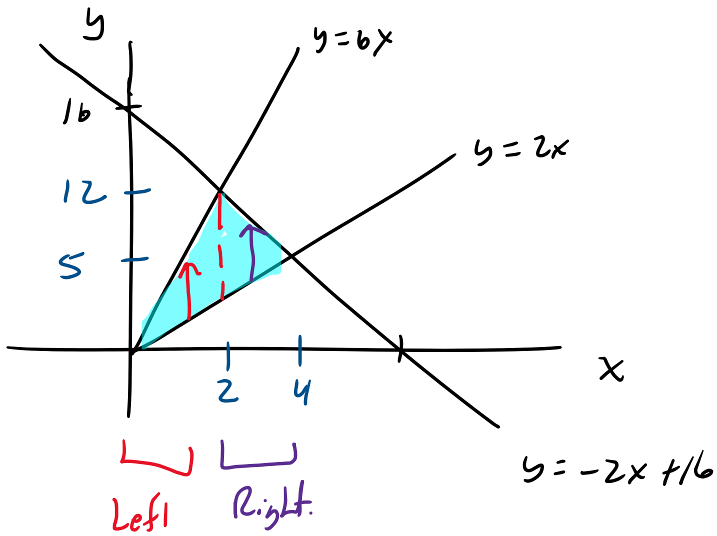
$\frac{1}{4}y \le x \le \sqrt{y}$

16

$$\iint_D x+y \, dA = \int_{y=0}^{16} \int_{x=\frac{1}{4}y}^{\sqrt{y}} x+y \, dx\, dy = \cdots = 89.6$$

Example: Set up the double integral that would evaluate the function $f(x, y) = x + y$ over the region bounded by the lines $y = 6x$, $y = 2x$, and $y = -2x + 16$

$$\iint_D x+y \, dA$$



$dy\,dx \uparrow$

Left

$0 \le x \le 2$

$2x \le y \le 6x$

Right

$2 \le x \le 4$

$2x \le y \le -2x+16$

$$\iint_D x+y \, dA = \int_{x=0}^{2} \int_{y=2x}^{6x} x+y \, dy\,dx + \int_{x=2}^{4} \int_{y=2x}^{-2x+16} x+y \, dy\,dx$$

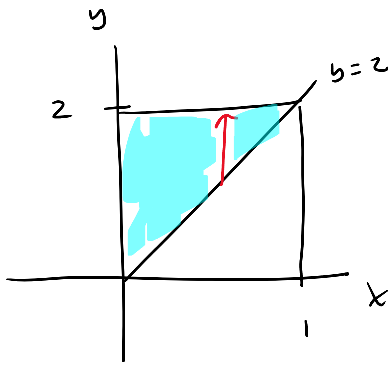
Example: Evaluate the integral by changing the order of integration $\int_0^1 \int_{2x}^2 \cos(y^2)\,dy\,dx$

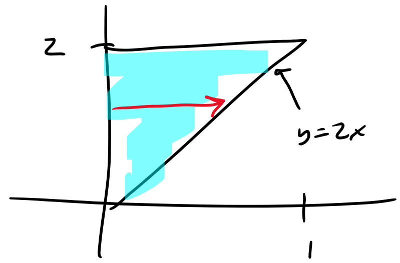
$x=$ $y=$

$dy\,dx$ ↑

$0 \le x \le 1$

$\underbrace{2x \le y}_{y=2x} \underbrace{\le 2}_{y=2}$





$dx\,dy$

$0 \le y \le 2$

$0 \le x \le \frac{y}{2}$

$$\int_{x=0}^{1} \int_{y=2x}^{2} \cos(y^2)\,dy\,dx = \int_{y=0}^{2} \int_{x=0}^{\frac{1}{2}y} \cos(y^2)\,dx\,dy = \int_{y=0}^{2} x\cos(y^2)\Big|_{x=0}^{\frac{1}{2}y}\,dy$$

$$= \int_{y=0}^{2} \frac{1}{2}y\cos(y^2)\,dy = \frac{1}{4}\sin(y^2)\Big|_{y=0}^{2} = \frac{1}{4}\sin(4) - \frac{1}{4}\sin(0)$$

$u = y^2$

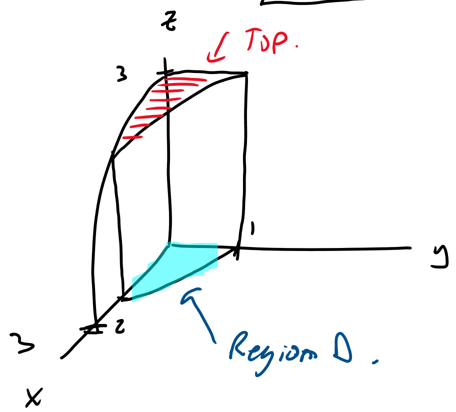
$$= \frac{1}{4}\sin(4)$$

Example Find the volume of the solid bounded by the cylinder $x^2 + z^2 = 9$, the planes $x = 0$, $y = 0$, $z = 0$, $\underbrace{x + 2y = 2}$ in the first octant.

$x = 0$ yz plane

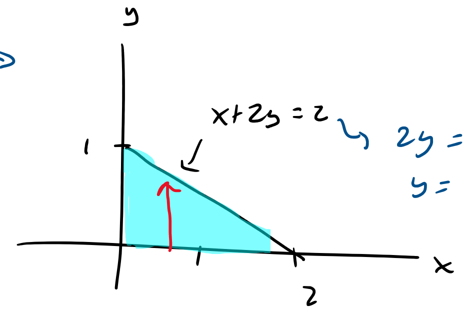
$y = 0$ xz plane

$z = 0$ xy plane



Top: $z = \sqrt{9 - x^2}$

Region D $\longrightarrow$



$\hookrightarrow 2y = 2 - x$

$y = 1 - \frac{x}{2}$

$\uparrow dy\,dx$ $\qquad dx\,dy \rightarrow$

$0 \le x \le 2$

$0 \le y \le 1 - \frac{x}{2}$

$$V = \iint_D \sqrt{9 - x^2}\, dA$$

$$= \int_{x=0}^{2} \int_{y=0}^{1 - \frac{x}{2}} \sqrt{9 - x^2}\, dy\, dx$$

$$= \int_{x=0}^{2} y\sqrt{9 - x^2}\,\Big|_{y=0}^{1 - \frac{x}{2}} dx = \int_{x=0}^{2} \left(1 - \frac{x}{2}\right)\sqrt{9 - x^2}\, dx$$

$$= \underbrace{\int_{x=0}^{2} \sqrt{9 - x^2}\, dx}_{\substack{x = 3\sin\theta \\ \text{Trig sub}}} - \underbrace{\int_{x=0}^{2} \frac{1}{2} x \sqrt{9 - x^2}\, dx}_{\text{u-sub}}$$

$u = 9 - x^2$

$\vdots$

$$= \frac{1}{6}\left(11\sqrt{5} - 27\right) + \frac{9}{2}\arcsin\left(\frac{2}{3}\right)$$

Example: Setup the integral(s) that would give the volumn of the solid (a tetrahedron) bounded by the planes $x = 0$, $z = 0$, $x = 3y$ and $\underbrace{x + y + z = 4}$

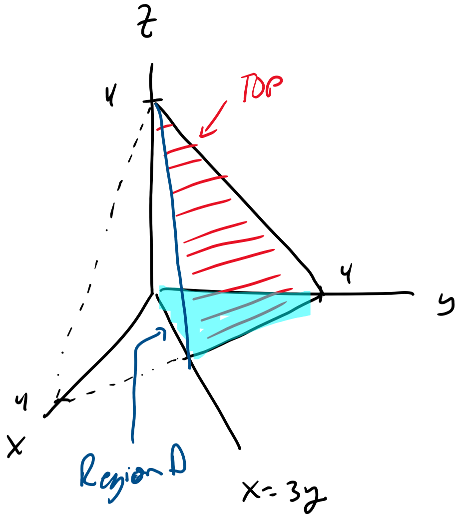
$x+y=4$

$3y+y=4$

$4y=4$

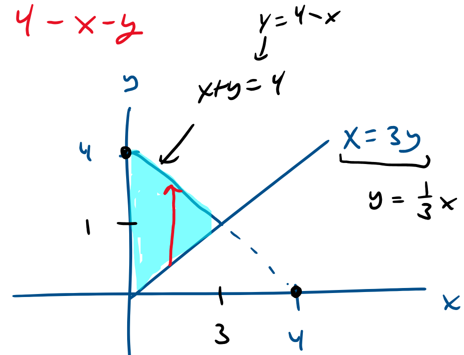
$y=1$

$x=3$



Top: $z = 4 - x - y$

Region D



$\uparrow$ $dy\,dx$

$0 \le x \le 3$

$\frac{1}{3}x \le y \le 4-x$

$$\iint_D 4-x-y \, dA = \int_{x=0}^{3} \int_{y=\frac{1}{3}x}^{4-x} 4-x-y \, dy \, dx$$

Example: Evaluate $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$

$\overrightarrow{dx\,dy}$

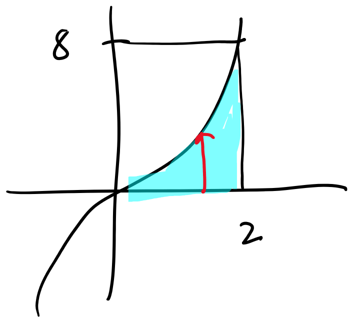
$0 \le y \le 8$

$\sqrt[3]{y} \le x \le 2$

$x = 2$

$\sqrt[3]{y} = x$

$y = x^3$



$\uparrow dy\,dx$

$0 \le x \le 2$

$0 \le y \le x^3$

$$\int_{x=0}^{2}\int_{y=0}^{x^3} e^{x^4}\, dy\, dx = \int_{x=0}^{2} y e^{x^4}\Big|_{y=0}^{x^3} dx$$

$$= \int_{x=0}^{2} x^3 e^{x^4}\, dx = \frac{1}{4} e^{x^4}\Big|_{x=0}^{2}$$

$u = x^4$

$$= \frac{1}{4} e^{16} - \frac{1}{4} e^{0}$$

$$= \frac{1}{4}\left(e^{16} - 1\right)$$

Properties of Double Integrals:

- If a is a real number then $\displaystyle\iint_D (af(x,y)+g(x,y))dA = a\iint_D f(x,y)dA + \iint_D g(x,y)dA$
- If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps on their boundaries then
  $$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$
- $\displaystyle\iint_D 1\ dA = A(D) =$ the area of region D.