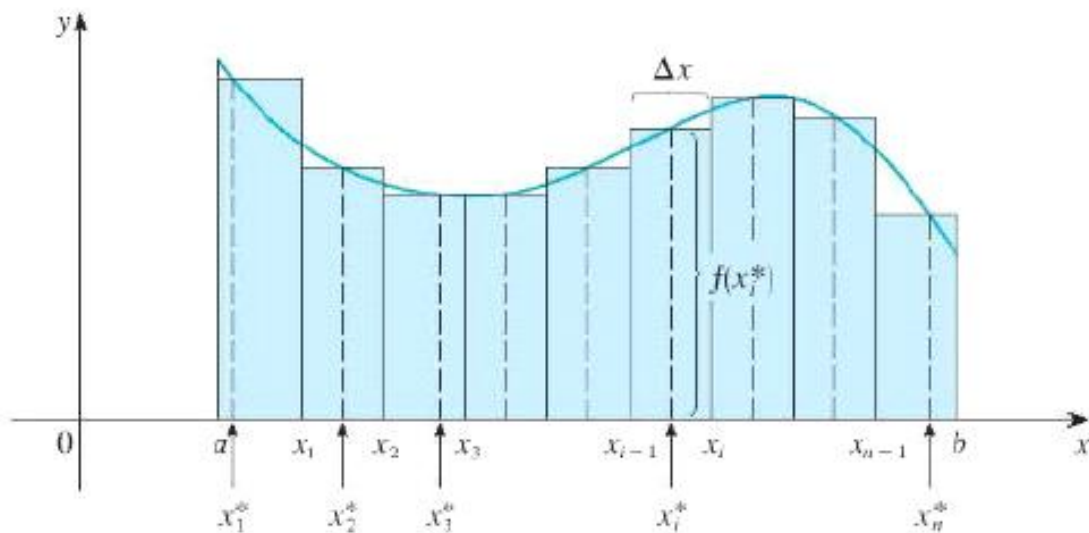


Section 15.1: Double Integrals over Rectangles

For a function of one variable, we define the integral in the following fashion. If $f(x)$ is defined on a closed interval $[a, b]$, we partition the interval $[a, b]$ into n equally spaced subintervals. Then

$$= \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \underline{\underline{f(x_i^*) \Delta x}}$$

If $f(x) \geq 0$ then this integral represent area.

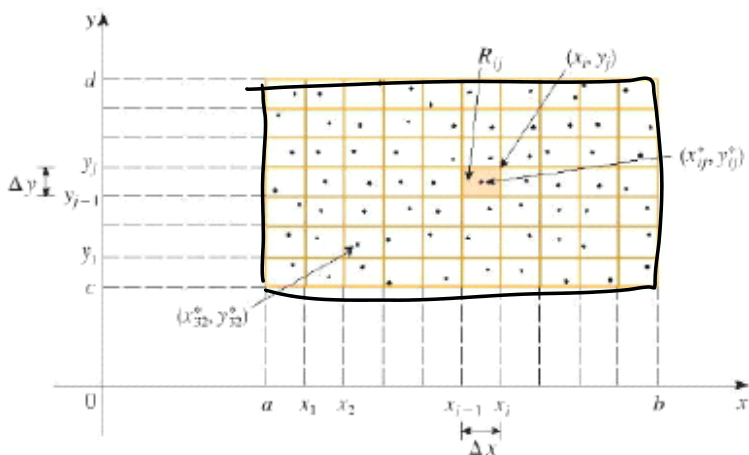


Definition of the double integral

Let $f(x, y)$ be a function of two variable defined on a closed rectangle R . Assume (for now) that $f(x, y) \geq 0$.

Define $R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$.

Lets partition both of the intervals $[a, b]$ into n equally spaced subintervals and $[c, d]$ into m equally spaced subintervals. This converts the rectangular region into a grid as shown in the picture.



Now choose a point (x_i^*, y_j^*) in the region R_{ij} . Then the **double integral** of f over the rectangle r is

$$\iint_R f(x, y) dA = \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m \underbrace{f(x_{ij}^*, y_{ij}^*) \Delta A}_{\text{area of subrectangle}}$$

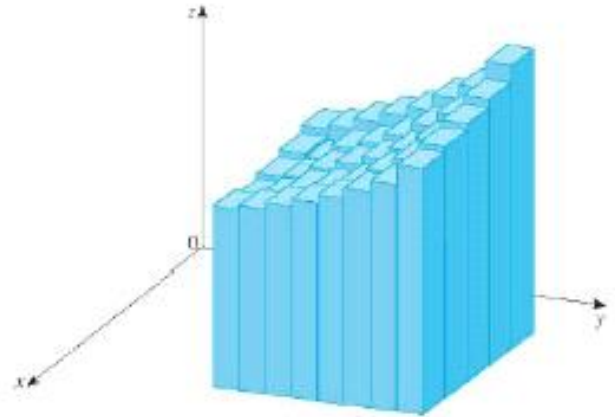
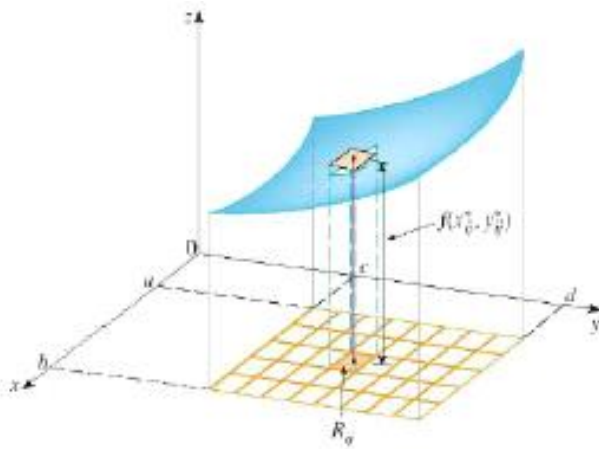
if this limit exists.

$dA = dy dx = dx dy$
for a rectangular box

Note: Another notation is $\iint_R f(x, y) dA = \iint_R f(x, y) dx dy$

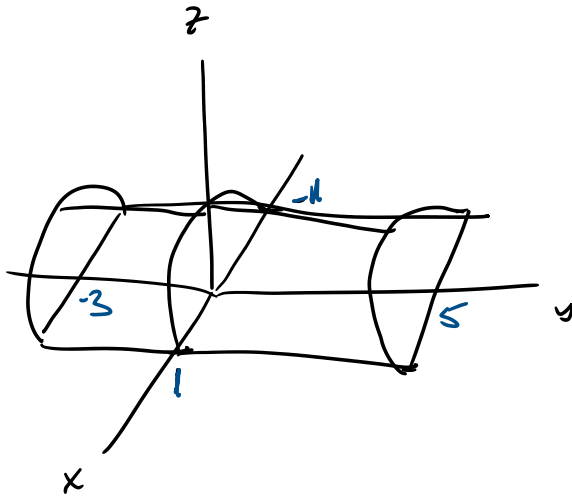
Theorem: If $f(x, y) \geq 0$ and f is continuous on the rectangle R , then the volume V of the solid S that lies above R and under the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y) \, dA$$



Example: Evaluate the integral $\iint_R \sqrt{1-x^2} dA$

where $R = [-1, 1] \times [-3, 5]$ by identifying it as a volume of a solid.



$$f(x, y) = \sqrt{1-x^2}$$

$$z = \sqrt{1-x^2}$$

$$z^2 = 1-x^2$$

$$x^2 + z^2 = 1 \quad R=1$$

(length) $\Rightarrow h =$

$$\iint_R \sqrt{1-x^2} dA = \frac{1}{2} \pi (1)^2 (8)$$

$$= 4\pi$$

Iterated Integrals

Definition: Suppose f is a function of two variables that is integrable over the rectangle

$$R = [a, b] \times [c, d].$$

x y

The partial integration of f with respect to x is $\int_a^b f(x, y) dx$ and means that y is held fixed and f is integrated with respect to x . This result is a function of the

variable y : $A(y) = \int_a^b f(x, y) dx$.

If we now integrate A with respect to the variable y we get

$$\int_c^d A(y) dy = \int_{y=c}^d \left[\int_{x=a}^b f(x, y) dx \right] dy = \int_{y=c}^d \int_{x=a}^b f(x, y) dx dy$$

The integral above is called an **iterated integral**. Similarly, we can get the following.

$$\int_{x=a}^b \left[\int_{y=c}^d f(x, y) dy \right] dx = \int_{x=a}^b \int_{y=c}^d f(x, y) dy dx$$

$$R = [0, 3] \times [1, 2]$$

Example: Evaluate the following.

$$\begin{aligned} \text{(A)} \int_{x=0}^3 \int_{y=1}^2 6x^2y \, dy \, dx &= \int_{x=0}^3 \left. \frac{6x^2y^2}{2} \right|_{y=1}^2 dx = \int_{x=0}^3 3x^2y^2 \Big|_{y=1}^2 dx \\ &= \int_{x=0}^3 (3x^2(4) - 3x^2(1)) dx = \int_{x=0}^3 (12x^2 - 3x^2) dx = \int_{x=0}^3 9x^2 dx \\ &= \left. \frac{9x^3}{3} \right|_{x=0}^3 = 3x^3 \Big|_0^3 = 3(3)^3 - 0 = 81 \end{aligned}$$

$$\begin{aligned} \text{(B)} \int_{y=1}^2 \int_{x=0}^3 6x^2y \, dx \, dy &= \int_{y=1}^2 \left. 2x^3y \right|_{x=0}^3 dy = \int_{y=1}^2 2(3^3y - 0) dy \\ &= \int_{y=1}^2 54y \, dy = \left. \frac{54y^2}{2} \right|_{y=1}^2 = 27y^2 \Big|_{y=1}^2 \\ &= 27(4) - 27(1) = 108 - 27 = 81 \end{aligned}$$

Fubini's Theorem: If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

In the case where $f(x, y) = g(x)h(y)$ then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d g(x)h(y) dy dx = \int_a^b g(x) dx \int_c^d h(y) dy$$

$$\begin{aligned} f(x, y) &= 6x^2 y \\ &= 3x^2 \cdot 2y \end{aligned}$$

$$\int_{x=0}^3 \int_{y=1}^2 6x^2 y dy dx = \int_{x=0}^3 3x^2 dx \cdot \int_{y=1}^2 2y dy = \dots = 81$$

Example: If $R = [1, 8] \times [0, \pi]$, evaluate

$$\iint_R y \cos(xy) dA$$

$$= \int_{y=0}^{\pi} \int_{x=1}^8 y \cos(xy) dx dy = \int_{y=0}^{\pi} y \cdot \frac{1}{y} \sin(xy) \Big|_{x=1}^8 dy$$

$$= \int_{y=0}^{\pi} \sin(8y) - \sin(y) dy$$

$$= \left(-\frac{1}{8} \cos(8y) + \cos(y) \right) \Big|_{y=0}^{\pi}$$

$$= -\frac{1}{8} \cos(8\pi) + \cos(\pi) - \left(-\frac{1}{8} \cos(0) + \cos(0) \right)$$

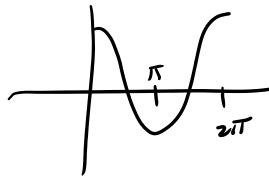
$$= -\frac{1}{8} (1) + (-1) + \frac{1}{8} - 1$$

$$= -2$$

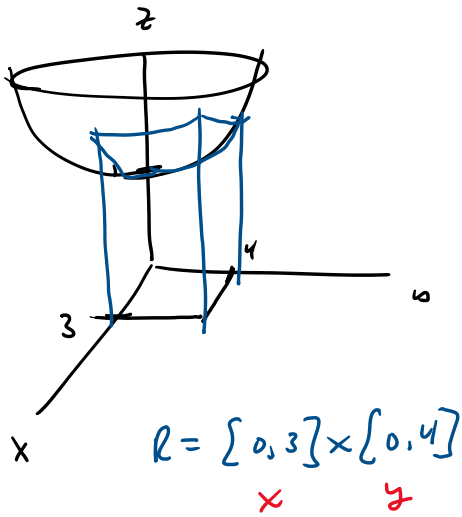
$dy dx$

Integration
by parts.

$dx dy$



Example: Find the volume of the solid S that is bounded by the elliptic paraboloid $z = 6x^2 + y^2 + 1$, the planes $x = 3$ and $y = 4$ and the three coordinate planes.



$$\iint_R z \, dA = \iint_R 6x^2 + y^2 + 1 \, dA$$

$$\int_{y=0}^4 \int_{x=0}^3 6x^2 + y^2 + 1 \, dx \, dy$$

$$= \int_{y=0}^4 \left. 2x^3 + xy^2 + x \right|_{x=0}^3 dy = \int_{y=0}^4 54 + 3y^2 + 3 \, dy$$

$$= \int_{y=0}^4 57 + 3y^2 \, dy = 57y + y^3 \Big|_{y=0}^4$$

$$= 57(4) + 4^3 - 0 = 228 + 64 = 292$$