Section 14.7: Maximum and Minimum values

Definition: A function of two variables has a **local maximum** at (a,b) if $f(x,y) \leq f(a,b)$ for all points (x,y) in some disk with center (a,b). The number $f(a,b)$ is called a **local maximum value.** If $f(x,y) \geq f(a,b)$ for all (x,y) in such disk, $f(a,b)$ is a **local minimum value.**

Note: The word local is sometimes replaced with the word relative.

Theorem: If f has a **local extremum** (that is, a local maximum or minimum) at (a,b) and the first-order partial derivatives of f exists there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$

Note: If the graph of f has a tangent plane at a local extremum, then the tangent plane is horizontal.

Definition: A point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or one of these partial derivatives does not exist, is called a **critical point** of f.

Second Derivative Test: Suppose the second partial derivatives of f are continuous in a disk with center (a, b), and suppose that (a, b) is a critical point of f. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

(a) If $D > 0$ and $f_{xx} > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx} < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is a sadle point.
(d) If $D = 0$ then the test gives no information.

$f_{xx} > 0$ ie. "concave up"

$f_{xx} < 0$ ie "concave down"

Example: Find and classify the critical values of

$f(x, y) = y^3 - 6y^2 - 2x^3 - 6x^2 + 48x + 20$

$f_x = -6x^2 - 12x + 48$

$f_y = 3y^2 - 12y$

$f_{xx} = -12x - 12$

$f_{yy} = 6y - 12$

$f_{xy} = 0$

$f_x = 0$

$-6x^2 - 12x + 48 = 0$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$x = -4, \quad x = 2$

$f_y = 0$

$3y^2 - 12y = 0$

$3y(y-4) = 0$

$y = 0 \quad y = 4$

points	$D = f_{xx} f_{yy} - (f_{xy})^2$	f_{xx}	conclusion
$(-4, 0)$	$(36)(-12) - (0)^2 < 0$		Saddle point
$(-4, 4)$	$(36)(12) - (0)^2 > 0$	positive	Local min
$(2, 0)$	$(-36)(-12) - (0)^2 > 0$	negative	Local max
$(2, 4)$	$(-36)(12) - (0)^2 < 0$		Saddle point

web assign

Local max value $= f(2,0)$

Local min value $= f(-4,4)$

value ie # not a point.

Saddle point $= (-4, 0, f(-4,0)), \; (2, 4, f(2,4))$

Example: Find and classify the critical values of $f(x,y) = x^3 + 6xy - 2y^2$

$f_x = 3x^2 + 6y$

$f_y = 6x - 4y$

$f_{xx} = 6x$

$f_{xy} = 6$

$f_{yy} = -4$

$f_x = 0$

$3x^2 + 6y = 0$

$3x^2 + 6\left(\frac{3x}{2}\right) = 0$

$3x^2 + 9x = 0$

$3x(x+3) = 0$

$x = 0$

$x = -3$

$f_y = 0$

$6x - 4y = 0$

$6x = 4y$

$\underbrace{y = \frac{6x}{4} = \frac{3x}{2}}$

if $\underline{x=0}$, $y = \frac{3(0)}{2} = 0$

$\underline{x=-3}$, $y = \frac{3(-3)}{2} = -\frac{9}{2}$

pts	$D = f_{xx} f_{yy} - (f_{xy})^2$	f_{xx}	conclusion
$(0,0)$	$(0)(-4) - (6)^2 < 0$		Saddle pt.
$(-3, -\frac{9}{2})$	$(-18)(-4) - (6)^2 > 0$	neg.	Local max

Example: Find and classify the critical values of $f(x, y) = 1 + 2xy - x^2 - y^2$

$f_x = 2y - 2x$

$f_y = 2x - 2y$

$f_{xx} = -2$

$f_{xy} = 2$

$f_{yy} = -2$

$f_x = 0$

$2y - 2x = 0$

$2y = 2x$

$y = x$

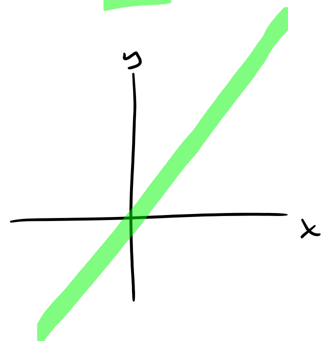
$f_y = 0$

$2x - 2y = 0$

$2x = 2y$

$x = y$

critical values are of the form (a, a) where a is a #



$$D = f_{xx} f_{yy} - (f_{xy})^2$$
$$= (-2)(-2) - (2)^2$$
$$= 0$$

$$f(x,y) = 1 + 2xy - x^2 - y^2 = 1 - (x^2 - 2xy + y^2)$$
$$= 1 - (x - y)^2$$

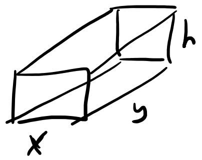
$x = y$ $\quad f(x,y) = 1$

$x \neq y$ $\quad f(x,y) < 1$

all critical values are a max

Example: The base of a rectangular tank with volume of 540 cubic units is made of slate and the sides are made of glass. If slate costs five times as much as glass(per unit area), find the dimensions of the tank which minimize the cost of the materials.

Since the slate is 5 times the cost of glass I chose the cost of glass to be \$1, and thus cost of slate is \$5



$V = xyh = 540$

$h = \dfrac{540}{xy}$

$C = \underbrace{\$5\, xy}_{\text{slate}} + \underbrace{\$1\,(2xh + 2yh)}_{\text{glass}}$

$C = 5xy + 2xh + 2yh$

$C = 5xy + \dfrac{1080}{y} + \dfrac{1080}{x}$

Solve volume formula to get the function into 2 variables.

first partials to find C.V.

$C_x = 5y - \dfrac{1080}{x^2} = 5y - 1080x^{-2}$

$C_y = 5x - \dfrac{1080}{y^2}$

$C_x = 0$

$5y - \dfrac{1080}{x^2} = 0$

$5y = \dfrac{1080}{x^2}$

$y = \dfrac{1080}{5x^2}$

$C_y = 0$

$5x - \dfrac{1080}{y^2} = 0$

$5x - \dfrac{1080}{\frac{(1080)^2}{25x^4}} = 0$

$5x - 1080 \cdot \dfrac{25x^4}{1080^2} = 0$

$5x - \dfrac{25x^4}{1080} = 0$

$5x\left(1 - \dfrac{5x^3}{1080}\right) = 0$

$x = 0$ not valid. Can not have a side of length zero

$1 - \dfrac{5x^3}{1080} = 0$

$1 = \dfrac{5x^3}{1080}$

$\dfrac{1080}{5} = x^3$

$x^3 = 216$

$x = 6$ ← only valid C.V.

2nd partials to test C.V. for max/min.

$C_{xx} = 2160x^{-3} = \dfrac{2160}{x^3}$

$C_{xx} > 0$ for any $x > 0$

$C_{yy} = \dfrac{2160}{y^3}$

$C_{xy} = 5$

$D = \dfrac{2160}{x^3} \cdot \dfrac{2160}{y^3} - 25$

$C_{xx} \cdot C_{yy} - (C_{xy})^2$

if $D > 0$ then the point is a min since

$x = 6$

$y = \dfrac{1080}{5(6^2)} = \dfrac{216}{6^2} = 6$

$h = \dfrac{540}{xy} = \dfrac{540}{6 \cdot 6} = 15$

$$h = \frac{540}{xy} = \frac{540}{6 \cdot 6} = 15$$

$x = 6$ $h = 15$
$y = 6$

$$D = \frac{2160}{6^3} \cdot \frac{2160}{6^3} - 25$$

$$= 10 \cdot 10 - 25 > 0$$

So the critical point is a min

Absolute Maximum and Absolute Minimum

Definition: A function f has a **absolute maximum** at (a, b) if $f(a, b)$ is the largest function value for the domain of f. Similarly, f has a **absolute minimum** at (a, b) if $f(a, b)$ is the smallest function value for the domain of f.

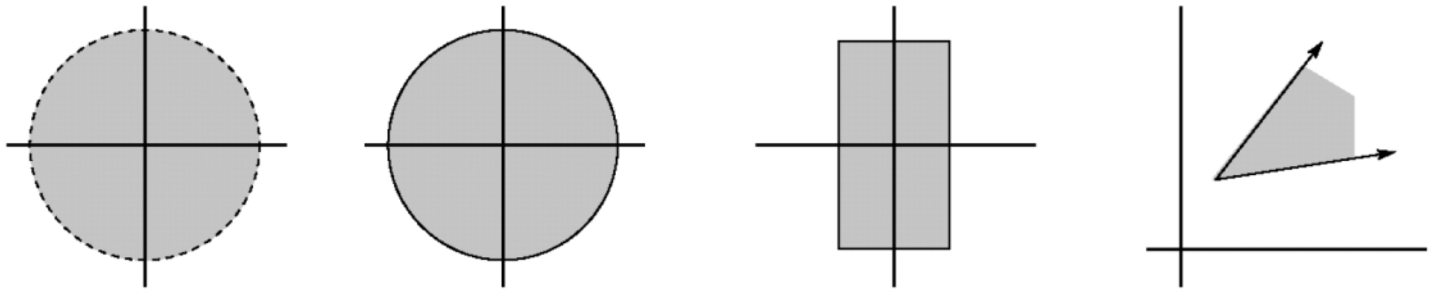
Extreme Value Theorem for Functions of One Variable: If f is continous on a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum value. These are found by evaluating critical points and the endpoints of the interval.

Extreme Value Theorem for Functions of Two Variables: If f is continuous on a closed and bounded set D in $\Re^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Definition: A closed set in $\Re^2$ is one that contains all of its boundry points.

Definition: A bounded set in $\Re^2$ is one that is contained in some disk.

$\{(x,y)|x^2+y^2<4\}$ $\{(x,y)|x^2+y^2\leq 4\}$ $\{(x,y)||x|\leq 1, |y|\leq 2\}$



not closed, Bounded | closed, Bounded | closed, Bounded | closed, not Bounded

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

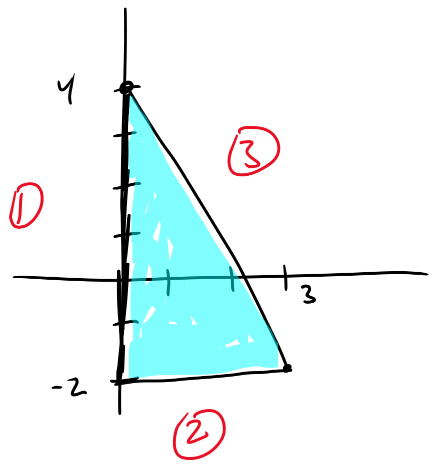
(1) Find the values of f at the critical points in D.
(2) Find the extreme values of f on the boundry of D.
(3) The largest of the values is the absolute maximum value; the smallest is the absolute minimum value.

Example: Find the absolute maximum/absolute minimum of f on the set D.

$D = \{(x,y) |\ 0 \le x \le 3,\quad -2 \le \underbrace{y \le 4-2x}\}$

$y = 4 - 2x$

$f(x,y) = x^2 + xy + 2y^2 - 3x + 2y$



$f_x = 2x + y - 3$

$f_y = x + 4y + 2$

$0 = 2x + y - 3$

$0 = x + 4y + 2$

$\underbrace{-4y - 2}= x$

$0 = 2(-4y-2) + y - 3$

$0 = -8y - 4 + y - 3$

$7y = -7$

$y = -1$

$x = -4(-1) - 2$

$= 4 - 2$

$x = 2$

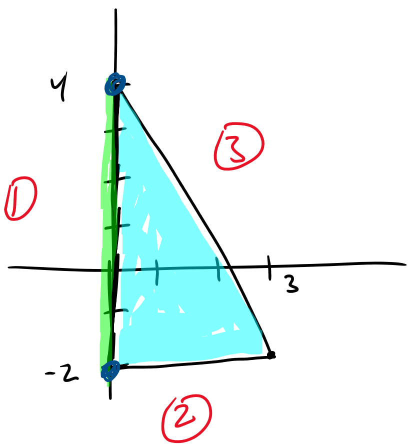
critical point. $(2, -1)$

Example: Find the absolute maximum/absolute minimum of f on the set D.

$D = \{(x,y) \mid 0 \le x \le 3, \ \ -2 \le \underbrace{y \le 4-2x}\}$

$y = 4-2x$

$f(x,y) = x^2 + xy + 2y^2 - 3x + 2y$



Boundary #1

$x = 0 \qquad -2 \le y \le 4$

$g(y) = f(0,y) = 0 + 0 + 2y^2 - 0 + 2y$

$g(y) = 2y^2 + 2y$

$g' = 4y + 2$

$0 = 4y + 2$

$-2 = 4y$

$y = -\frac{2}{4} = \frac{-1}{2}$

critical point

$(0, \frac{-1}{2})$

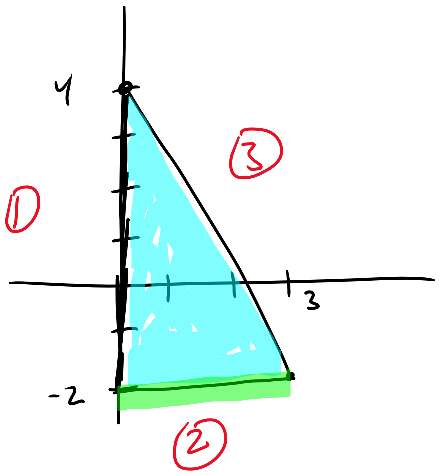
Example: Find the absolute maximum/absolute minimum of f on the set D.

$D = \{(x, y)|\ 0 \le x \le 3,\ \ -2 \le \underbrace{y \le 4 - 2x}\}$

$f(x, y) = x^2 + xy + 2y^2 - 3x + 2y$

$y = 4 - 2x$

Boundary #2



$0 \le x \le 3 \qquad y = -2$

$g(y) = f(x, -2) = x^2 - 2x + 8 - 3x - 4$

$g(y) = x^2 - 5x + 4$

$g' = 2x - 5$

$0 = 2x - 5$

$5 = 2x$

$x = \frac{5}{2}$

Critical point

$\left(\frac{5}{2}, -2\right)$

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Example: Find the absolute maximum/absolute minimum of f on the set D.

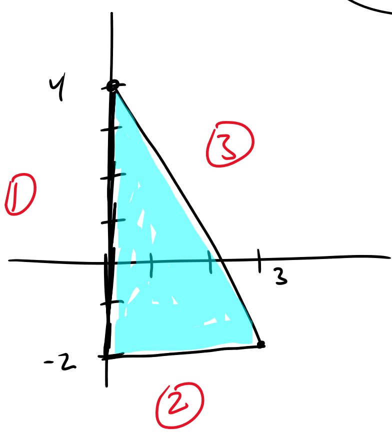
$D = \{(x,y) |\ 0 \le x \le 3,\ \ -2 \le \underbrace{y \le 4 - 2x}\}$

$y = 4 - 2x$

$f(x,y) = x^2 + xy + 2y^2 - 3x + 2y$

Boundary #3

$0 \le x \le 3 \qquad y = 4 - 2x$



$$g(x) = f(x, 4-2x) = x^2 + x(4-2x) + 2(4-2x)^2 - 3x + 2(4-2x)$$
$$= x^2 + 4x - 2x^2 + 2(16 - 16x + 4x^2) - 3x + 8 - 4x$$
$$= x^2 + 4x - 2x^2 + 32 - 32x + 8x^2 - 3x + 8 - 4x$$

$$g(x) = 7x^2 - 35x + 40$$

$$g'(x) = 14x - 35$$
$$0 = 14x - 35$$
$$35 = 14x$$

$$x = \frac{35}{14} = \frac{5}{2}$$

$$y = 4 - 2\left(\tfrac{5}{2}\right)$$
$$= 4 - 5$$
$$= -1$$

Critical point

$\left(\frac{5}{2}, -1\right)$

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Example: Find the absolute maximum/absolute minimum of f on the set D.

$D = \{(x, y) | \ 0 \le x \le 3, \quad -2 \le y \le 4 - 2x\}$

$f(x, y) = x^2 + xy + 2y^2 - 3x + 2y$

	points	f(x,y)	
critical point	(2,-1)	-4	Abs min
Corner points	(0,-2)	4	
	(0,4)	40	Abs max
	(3,-2)	2	
Boundary #1	$(0, -\frac{1}{2})$	-.5	
Boundary #2	(2.5, -2)	-2.25	
Boundary #3	(2.5, -1)	-3.75	

Abs max = 40
Location (0,4)

Abs min = -4
Location (2,-1)

Example: Find the absolute max for $f(x, y) = xy$ on the set D.

$$D = \left\{ (x, y) \middle| \frac{x^2}{16} + y^2 \leq 1 \right\}$$

We can solve $\frac{x^2}{16} + y^2 = 1$ for y. This gives $y = \pm\sqrt{1 - \frac{x^2}{16}}$

Thus our formula becomes

$$g(x) = f(x, y) = f\left(x, +\sqrt{1 - \frac{x^2}{16}}\right)$$

$$g(x) = x\sqrt{1 - \frac{x^2}{16}} \quad \text{(Top curve)}$$

or Bottom curve

$$g(x) = -x\sqrt{1 - \frac{x^2}{16}}$$

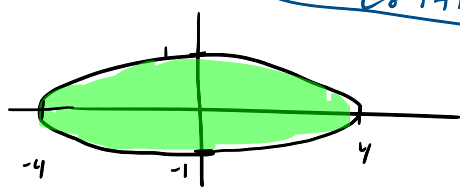
Now this is a 2 function boundary must do both parts to find critical values

or Boundary done in one step

$f_x = y$ $\qquad$ $f_y = x$

$f_x = 0$ $\qquad$ $f_y = 0$

$0 = y$ $\qquad$ $x = 0$



critical pt $(0,0)$

$\cos^2\theta + \sin^2\theta = 1$

$\cos\theta = \frac{x}{4}$ $\qquad$ $\sin\theta = y$

$x = 4\cos\theta$ $\qquad$ $y = \sin\theta$

$f(x,y) = xy = 4\cos\theta\sin\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

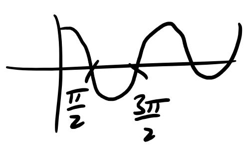
$f = 2\sin(2\theta)$

$f' = 2\cos(2\theta) \cdot 2$

$= 4\cos(2\theta)$

$0 = 4\cos(2\theta)$

$\cos(2\theta) = 0$



$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \ldots\ldots$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

	critical pt	$f(x,y) = xy$
	$(0,0)$	0
$\theta = \frac{\pi}{4}$	$\left(\frac{4\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	2
$\theta = \frac{3\pi}{4}$	$\left(-\frac{4\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	-2
$\theta = \frac{5\pi}{4}$	$\left(-\frac{4\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	2
$\theta = \frac{7\pi}{4}$	$\left(\frac{4\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	-2

note if the given function $f(x,y)$ was different

then we would work this in a different method.

ie. if $f(x,y) = xy^2$ Then just solve

the equation $\frac{x^2}{16} + y^2 = 1$ for y^2

and set $y^2 = 1 - \frac{x^2}{16}$

Then $g(x) = f(x,y) = x\left(1 - \frac{x^2}{16}\right)$ where $-4 \le x \le 4$

now use this function to find the critical values.