Section 14.6: Directional Derivatives and the Gradient Vector

Recall that for $f(x, y)$, the first partial f_x represent the rate of change of f in the x direction and f_y represents the rate of change of f in the y direction. In other words, f_x and f_y represent the rate of change of f in the direction of the unit vectors **i** and **j** respectively.

Definition: The **directional derivative** of f at (x_o, y_o) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$, denoted by $D_{\mathbf{u}} f(x_o, y_o)$, is

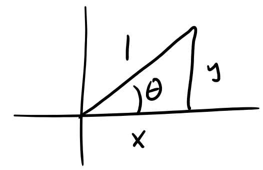
$$D_{\mathbf{u}} f(x_o, y_o) = \lim_{h \to 0} \frac{f(x_o + ha, y_o + hb) - f(x_o, y_o)}{h}$$

if this limit exists.

Note: This shows that $f_x(x_o, y_o) = D_{\mathbf{i}} f(x_o, y_o)$ and $f_y(x_o, y_o) = D_{\mathbf{j}} f(x_o, y_o)$.

Theorem: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$$



$\frac{x}{1} = \cos\theta$

$x = \cos\theta$

$y = \sin\theta$

Note: If the unit vector $\mathbf{u}$ makes an angle θ with respect to the positive $x-$axis, then we can write

$\mathbf{u} = \langle \cos\theta, \sin\theta\rangle$ and $D_{\mathbf{u}}f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$

Example: Find the rate of change of $f(x,y) = x^2 + 2xy - 3y^2$ at the point $(1,2)$ in the direction indicated by the angle $\theta = \frac{\pi}{4}$.

$u = \langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4} \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$f_x = 2x + 2y$

$f_x(1,2) = 2(1) + 2(2) = 6$

$f_y = 2x - 6y$

$f_y(1,2) = 2(1) - 6(2) = -10$

$D_u f(1,2) = f_x(1,2)\,\frac{\sqrt{2}}{2} + f_y(1,2)\cdot\frac{\sqrt{2}}{2}$

$= 6\frac{\sqrt{2}}{2} + -10\frac{\sqrt{2}}{2}$

$= 3\sqrt{2} - 5\sqrt{2}$

$= -2\sqrt{2}$

If we move 1 unit from the pt $(1,2)$ in the direction of vector u the function value $z=f(x,y)$ will decrease by approx $2\sqrt{2}$

$\nabla \Rightarrow$ nabla

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$

Definition: If f is a function of two variables x and y, then the **gradient** of f, denoted **grad** f or ∇f, is the vector function defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \langle f_x, f_y \rangle$$

Note: ∇f which is read "del f".

Theorem: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_\mathbf{u} f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x, f_y \rangle \cdot \langle a, b \rangle$$

$$D_u f(x,y) = \nabla f \cdot u$$

Definition: The gradient and the directional derivative of a function $f(x, y, z)$ with unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$ $\qquad$ $D_\mathbf{u} f(x, y, z) = \nabla f \cdot u$

Example: Find the gradient and the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of the vector $\mathbf{v} = \langle 2, 5\rangle$.

$\nabla f = \langle f_x, f_y \rangle$

$= \langle 2xy^3, 3x^2y^2 - 4 \rangle$

need unit vector

$|v| = \sqrt{4+25} = \sqrt{29}$

$\nabla f(2,-1) = \langle 2(2)(-1)^3, 3(2)^2(-1)^2 - 4 \rangle$

$= \langle -4, 12 - 4 \rangle$

$= \langle -4, 8 \rangle$

$u = \frac{1}{\sqrt{29}} \langle 2, 5 \rangle = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$

$D_u f(2,-1) = \nabla f \cdot u = \langle -4, 8 \rangle \cdot \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$

$= \frac{-8}{\sqrt{29}} + \frac{40}{\sqrt{29}} = \frac{32}{\sqrt{29}}$

Example: Find the directional derivative of the function $f(x, y, z) = z^4 - x^3y^2$ at the point $P(1, 3, 2)$ in the direction of $Q(2, 4, 3)$.

$$\nabla f = \langle f_x, f_y, f_z \rangle$$
$$= \langle -3x^2y^2, -2x^3y, 4z^3 \rangle$$

$$\nabla f(1,3,2) = \langle -27, -6, 32 \rangle$$

$$\vec{PQ} = \langle 2-1, 4-3, 3-2 \rangle$$
$$= \langle 1, 1, 1 \rangle$$

$$|\vec{PQ}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$u = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$$

$$D_u f(1,3,2) = \nabla f \cdot u = \langle -27, -6, 32 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$$
$$= \frac{1}{\sqrt{3}} \left[-27(1) + -6(1) + 32(1) \right]$$
$$= \frac{1}{\sqrt{3}} \left[-27 - 6 + 32 \right] = \frac{-1}{\sqrt{3}}$$

$$\vec{a} \cdot c\vec{b} = c(\vec{a} \cdot \vec{b})$$

The directional derivatives at a point P for a function f gives the rates of changes of f in all possible directions. This leads to the question: In which of these directions does f change the fastest and what is the maximum rate of change?

Theorem: Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occur when $\mathbf{u}$ has the direction as the gradient vector.

$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|u| \cos\theta$ $\quad$ θ is the angle between ∇f and u

$= |\nabla f| \cos\theta$

$\cos\theta = 1$ when $\theta = 0$ ie $\cos\theta$ has max.

max rate of change is $|\nabla f|$

which direction is the max rate of change? ∇f.

min rate of change is when $\theta = \pi$ i.e. min is $-|\nabla f|$

direction of the min is $-\nabla f$.

Example: If $f(x, y) = xe^y$, in what direction does f have the maximum rate of change at the point $P(2, 0)$? What is the maximum rate of change of f?

$$\nabla f = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

direction

$$\nabla f(2,0) = \langle e^0, 2e^0 \rangle = \langle 1, 2 \rangle$$

max Rate of change is $|\nabla f(2,0)| = \sqrt{1^2 + 2^2} = \sqrt{5}$

Tangent Planes to Level surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. Let $P(x_o, y_o, z_o)$ be a point on the surface.

Let C be any curve on the surface going though the point P and defined by the vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ with $\mathbf{r}(t_o) = \langle x_o, y_o, z_o\rangle$.

Combining the above information gives $F(x(t), y(t), z(t)) = k$. Now if F and $\mathbf{r}$ are differentiable, then we can use the chain rule to get the following.

$$F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt} = 0$$

$$\langle F_x, F_y, F_z\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle = 0$$

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

Thus ∇F at point P is perpendicular to the tangent vector $\mathbf{r}'(t_o)$ to any curve C on the surface S passing though the point P.

This means that ∇F at point P is a normal vector for the tangent plane.

Level surface

$F(x,y,z) = k$ at pt P.

$n = \nabla F = \langle F_x(pt), F_y(pt), F_z(pt)\rangle$

is a normal vector to the tangent plane.

$z = f(x,y)$

$n = \langle -f_x, -f_y, 1\rangle$

$n = \langle f_x, f_y, -1\rangle$

Thus the tangent plane to the surface $F(x, y, z) = k$ at point $P(x_o, y_o, z_o)$ is

$$\underbrace{F_x(x_o, y_o, z_o)(x - x_o) + F_y(x_o, y_o, z_o)(y - y_o) + F_z(x_o, y_o, z_o)(z - z_o) = 0}$$

What is a direction vector for the normal line to the surface at the point $P(x_o, y_o, z_o)$?

$\hookrightarrow \nabla F(\text{pt.})$

$$x = x_o + t\, F_x(\text{pt})$$
$$y = y_o + t\, F_y(\text{pt})$$
$$z = z_o + t\, F_z(\text{pt})$$

Example: Find the equation of the tangent plane and the normal line to the surface at the point $(1, 2, 1)$

$$z = +\sqrt{\frac{17 - 4x^2 - y^2}{9}}$$

$$\underbrace{4x^2 + y^2 + 9z^2}_{F(x,y,z)} = 17$$

$\nabla F = \langle 8x, 2y, 18z \rangle$

$n = \nabla F(1,2,1) = \langle 8, 4, 18 \rangle$

$\boxed{8(x-1) + 4(y-2) + 18(z-1) = 0}$ ← tangent plane

normal line

$$\boxed{\begin{aligned} x &= 1 + 8t \\ y &= 2 + 4t \\ z &= 1 + 18t \end{aligned}}$$

In a similar manner, the gradient vector $\nabla F(x_o, y_o)$ is perpendicular to the level curves $f(x,y) = k$ at the point (x_o, y_o).

Consider the topographical map of a hill and let $f(x,y)$ represent the elevation at the point (x,y). Draw a curve of steepest ascent.