Section 14.5: Chain Rule

Chain rule for functions of a single variable: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Example: Let $z = x^y$ with $x = t^3$ and $y = \sin(t)$. Compute $z'(t) = \dfrac{dz}{dt}$.

$$z = (t^3)^{\sin t}$$

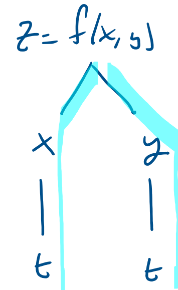
$$z = t^{3\sin t}$$

$$\ln z = \ln\left(t^{3\sin t}\right)$$

$$\ln(z) = 3\sin(t)\ln(t)$$

$$\frac{1}{z}\frac{dz}{dt} = 3\cos(t)\ln(t) + 3\sin(t)\cdot\frac{1}{t}$$

$$\frac{dz}{dt} = z\left[3\cos t\ln(t) + \frac{3\sin(t)}{t}\right] = t^{3\sin t}\left[3\cos t\ln(t) + \frac{3\sin t}{t}\right]$$



Chain Rule (Case 1): Suppose the $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad \text{or} \qquad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

Cal 1 Rule

$y = a^{f(x)}$

$y' = a^{f(x)} f'(x) \ln(a)$

$y = e^{x^2}$

$y' = 2x e^{x^2} \cdot \ln(e)$

Example: Let $z = x^y$ with $x = t^3$ and $y = \sin(t)$. Compute $z'(t) = \dfrac{dz}{dt}$

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

$$\frac{dz}{dt} = y x^{y-1} \cdot (3t^2) + x^y \cdot (1) \ln(x) (\cos(t))$$

Example: Let $z = \ln(x + y^2)$ with $x = \sqrt{1+t^2}$ and $y = e^{3t}$. Compute $z'(t)$

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

$$= \frac{1}{x+y^2} \cdot \left(\frac{1}{2}(1+t^2)^{-1/2} \cdot 2t \right) + \frac{2y}{x+y^2} \cdot 3e^{3t}$$

Example: The radius of a right circular cone is increasing at a rate of 2.1 in/s while its height is decreasing at a rate of 1.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in and the height is 140 in?

$\frac{dr}{dt} = 2.1$ in/sec

$\frac{dh}{dt} = -1.5$ in/sec

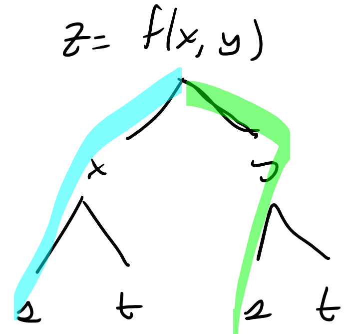
find $\frac{dV}{dt}$ when $r = 120$ in, $h = 140$ in

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = V_r \frac{dr}{dt} + V_h \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt} = \frac{2}{3}\pi(120)(140)(2.1) + \frac{1}{3}\pi(120)^2(-1.5)$$

$$\frac{dV}{dt} = 16320\pi \text{ cubic in/sec}$$



Chain Rule (Case 2): Suppose the $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ and all first partials of x and y exists. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$z_s = z_x \cdot x_s + z_y \cdot y_s$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: $z = \sin(x)\cos(y)$, where $x = (s-t)^2$ and $y = s^2 - t^2$. Compute z_s and z_t.

$$z_s = z_x \cdot x_s + z_y \cdot y_s$$

$$z_s = \cos(x)\cos(y) \cdot 2(s-t)^1(1) + (-\sin(x)\sin(y)) \cdot (2s)$$

$$z_t = z_x\, x_t + z_y \cdot y_t$$

$$= \cos(x)\cos(y) \cdot 2(s-t)^1(-1) + (-\sin(x)\sin(y)) \cdot (-2t)$$

$x = 1(2)e^0 = 2$

$y = 1(0)^3 + 2^2 = 4$

$z = 1(2)\sin(0) = 0$

Example: If $u = x^2y + y^3z^2$ where $x = rse^t$, $y = rt^3 + s^2$ and $z = rs\sin(t)$, find u_s when $(r, s, t) = (1, 2, 0)$.

	$r=1$, $s=2$, $t=0$
$u_x = 2xy$	$u_x = 2(2)(4) = 16$
$u_y = x^2 + 3y^2z^2$	$u_y = 2^2 + 3(4)^2(0) = 4$
$u_z = 2y^3z$	$u_z = 2(4)^3(0) = 0$
$x_s = re^t$	$x_s = 1e^0 = 1$
$y_s = 2s$	$y_s = 2(2) = 4$
$z_s = r\sin(t)$	$z_s = 1\sin(0) = 0$

$u_s = u_x x_s + u_y y_s + u_z z_s$

$= 16(1) + 4(4) + 0(0)$

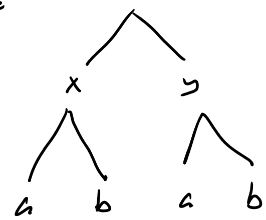
$= 16 + 16$

$= 32$

Example: Suppose f is a differentiable function of x and y, and $g(a,b) = f(e^a + \sin(b), e^a + \cos(b))$. Use the table of values to calculate $g_a(0,0)$ and $g_b(0,0)$.

	f	g	f_x	f_y
$(0,0)$	4	5	10	20
$(1,2)$	8	9	7	6

$g(a,b) = f(x,y)$



$x = e^a + \sin(b)$

$y = e^a + \cos(b)$

$x = e^0 + \sin(0) = 1$

$y = e^0 + \cos(0) = 2$

$g_a(0,0) \rightarrow$ $a=0 \quad b=0$

$g_a = f_x \cdot x_a + f_y \, y_a$

$g_a(0,0) = f_x(1,2)\, x_a(0,0) + f_y(1,2)\, y_a(0,0)$

$= 7 \cdot 1 + 6 \cdot 1 = 13$

$x_a = e^a$

$x_b = \cos(b)$

$y_a = e^a$

$y_b = -\sin(b)$

$a=0 \quad b=0$

$x_a = e^0 = 1$

$x_b = \cos(0) = 1$

$y_a = e^0 = 1$

$y_b = -\sin(0) = 0$

$g_b(0,0) = f_x(1,2)\, x_b(0,0) + f_y(1,2)\, y_b(0,0)$

$= 7(1) + 6(0) = 7$

$$\underbrace{x^2 + y^2 + e^{xy} + 7x}_{F(x,y)} = 0$$

Implicit Differentiation: Suppose that an equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x. i.e. $y = f(x)$ and $F(x, f(x)) = 0$ for all x in the domain of $f(x)$. Find $\frac{dy}{dx}$.

$$F(x,y) = 0$$

Chain Rule.

$$F_x \underbrace{\frac{dx}{dx}}_{1} + F_y \frac{dy}{dx} = 0$$

$$F_x + F_y \frac{dy}{dx} = 0$$

$$F_y \frac{dy}{dx} = -F_x$$

$$\boxed{\frac{dy}{dx} = \frac{-F_x}{F_y}}$$

Example: Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6x^2y^4$

$$\underbrace{x^3 + y^3 - 6x^2y^4}_{F(x,y)} = 0$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-\left(3x^2 - 12xy^4\right)}{3y^2 - 24x^2y^3}$$

Example: Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation $F(x, y, z) = 0$. Find z_x and z_y.

find z_x

$$F_x \underbrace{\frac{\partial x}{\partial x}}_{1} + F_y \underbrace{\frac{\partial y}{\partial x}}_{0} + F_z \frac{\partial z}{\partial x} = 0$$

$$F_x + F_z \frac{\partial z}{\partial x} = 0$$

$$z_x = \frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

$$z_y = \frac{-F_y}{F_z}$$

$$z_x = \frac{-F_x}{F_z}$$

$$\underbrace{x^4 + y^3 + z^2 + xye^z - 10}_{F(x,y,z)} = 0$$

Example: If $\underbrace{x^4 + y^3 + z^2 + xye^z = 10}$. Find

(a) z_x

$$z_x = \frac{-F_x}{F_z} = \frac{-(4x^3 + ye^z)}{2z + xye^z}$$

(b) x_y $= \dfrac{-F_y}{F_x} = \dfrac{-(3y^3 + xe^z)}{4x^3 + ye^z}$