Section 13.2: Derivatives and Integrals of Vector Functions

Theorem Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable

functions, then $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$

$r(t)$

Note: If $\mathbf{r}(t)$ is a **position function** of a particle at time t, then the **velocity function** is $\mathbf{r}'(t) = \mathbf{v}(t)$ and the **acceleration function** is $\mathbf{r}''(t) = \mathbf{v}'(t) = \mathbf{a}(t)$.

$r'(t) =$ tangent vector.

Definition: The **unit tangent vector** at t is defined to be $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

Theorem Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then.

$$\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t)) \quad \text{chain rule}$$

Example: Given $\mathbf{r}(t) = \langle 3t,\ e^{2t-4},\ \sin(t\pi)\rangle$.

(a) Find a tangent vector to the curve at $t = 0$.

$$r'(t) = \langle 3,\ 2e^{2t-4},\ \pi\cos(t\pi)\rangle$$

$$r'(0) = \langle 3,\ 2e^{-4},\ \pi\cos(0)\rangle = \langle 3,\ 2e^{-4},\ \pi\rangle$$

(b) Find $\mathbf{T}(0)$. $= \dfrac{r'(0)}{|r'(0)|} =$

$$|r'(0)| = \sqrt{3^2 + (2e^{-4})^2 + (\pi)^2} = \sqrt{9 + 4e^{-8} + \pi^2}$$

$$= \frac{1}{\sqrt{9 + 4e^{-8} + \pi^2}}\langle 3,\ 2e^{-4},\ \pi\rangle$$

$$T(t) = \frac{1}{\sqrt{3^2 + (2e^{2t-4})^2 + (\pi\cos(t\pi))^2}}\langle 3,\ 2e^{2t-4},\ \pi\cos(t\pi)\rangle$$

$t=0$

(c) Find a tangent line to the curve at the point $(6, 1, 0)$ $\rightarrow$ $t = 2$

$3t = 6$
$t = 2$

direction vector = tangent vector = $r'(2) = \langle 3,\ 2e^{0},\ \pi\cos(2\pi)\rangle$
$= \langle 3,\ 2,\ \pi\rangle$

tangent line $= \langle 6 + 3t,\ 1 + 2t,\ 0 + \pi t\rangle$

$r = \langle a, b, c \rangle$

$r \cdot r = a^2 + b^2 + c^2$

Example: Show that if $\underbrace{|\mathbf{r}(t)| = c}$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$

$r \cdot r = |r(t)|^2 = c^2$

$\frac{d}{dt} r \cdot r = \frac{d}{dt} c^2$

$r' \cdot r + r \cdot r' = 0$

$2(r \cdot r') = 0$

$r \cdot r' = 0$

Example: Given $\mathbf{r}(t) = \langle 3t,\ e^{2t-4},\ \sin(t\pi)\rangle$. compute $\int \mathbf{r}(t)\ dt$

$$\int r(t)\,dt = \left\langle \int 3t\,dt,\ \int e^{2t-4}\,dt,\ \int \sin(t\pi)\,dt \right\rangle$$

$$= \left\langle \frac{3t^2}{2} + C_1,\ \frac{1}{2}e^{2t-4} + C_2,\ \frac{-1}{\pi}\cos(t\pi) + C_3 \right\rangle$$

$$= \left\langle \frac{3t^2}{2},\ \frac{1}{2}e^{2t-4},\ \frac{-1}{\pi}\cos(t\pi) \right\rangle + \langle C_1, C_2, C_3 \rangle$$

$$= \left\langle \frac{3t^2}{2},\ \frac{1}{2}e^{2t-4},\ \frac{-1}{\pi}\cos(t\pi) \right\rangle + C$$

where C is a constant vector.