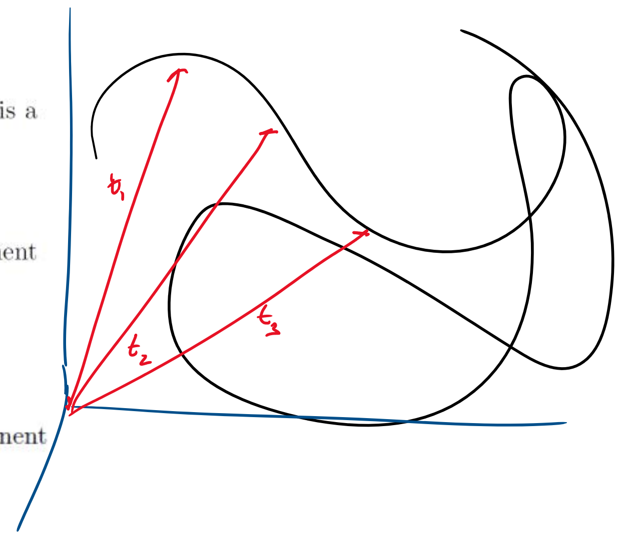
Section 13.1: Vector Functions and Space curves



Let $\mathbf{r}$ be a **vector function** whose domain is a set of real numbers and result is a three-dimensional vector. Let

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where $f(t)$, $g(t)$, and $h(t)$ are real valued functions and are called the component functions of $\mathbf{r}$.

The limit of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions:

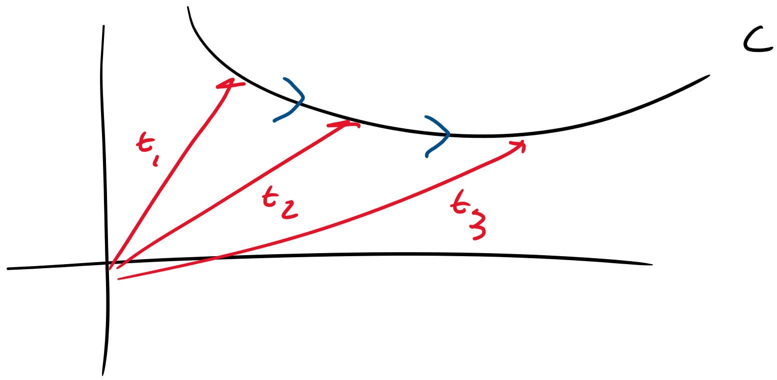
$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

A vector function $\mathbf{r}$ is continuous if and only if its component functions $f(t)$, $g(t)$, and $h(t)$ are continuous.

Definition: Suppose that $f(t)$, $g(t)$, and $h(t)$ are real valued functions on an interval I, then the set C defined as :

$$C = \{(x, y, z) | x = f(t), y = g(t), z = h(t)\}$$

where t is a parameter and t varies in some interval, I, is called a space curve. The space curve C can be traversed by the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$.



if $t_1 < t_2 < t_3$

Circular paraboloid.

x y z

Example: Does the space curve $\mathbf{r}(t) = \langle t, 0, 2t - t^2 \rangle$ lie on the paraboloid $z = x^2 + y^2$?
Does it intersect the parabaloid? yes

$x = t$

$y = 0$

$z = 2t - t^2$

$2t - t^2 = (t)^2 + (0)^2$

$2t - t^2 = t^2$

$2t = 2t^2$

not true for all values of t so space curve is not on the paraboloid.

Is the point $(1, 1, 7)$ on the paraboloid? no

$t = 1$

$\frac{2t}{t} = \frac{2t^2}{t}$

$2 = 2t$

$1 = t$

$2t = 2t^2$

$0 = 2t^2 - 2t$

$0 = 2t(t-1)$

$t = 0 \quad t = 1$

Intersects at $t = 0$ & $t = 1$

points

$t = 0$	$t = 1$
$x = 0$	$x = 1$
$y = 0$	$y = 0$
$z = 0$	$z = 1$

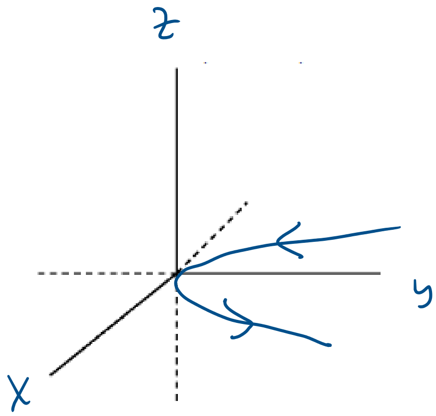
Example: Describe the curve defined by the vector function. Indicate the direction of motion.

$x \quad y \quad z$

(a) $\mathbf{r}(t) = \langle t, t^2, 0 \rangle$

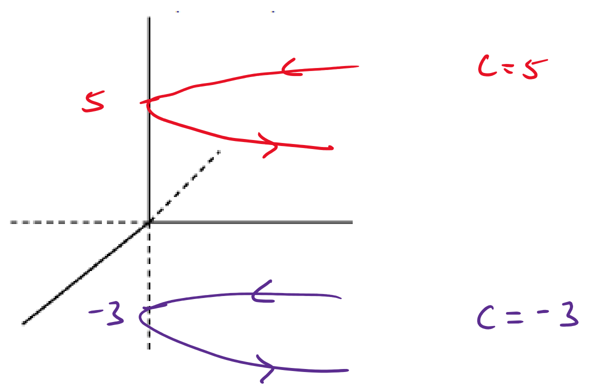
$\left.\begin{array}{l} x = t \\ y = t^2 \end{array}\right\} \rightarrow y = x^2$

$z = 0$



$y = x^2$

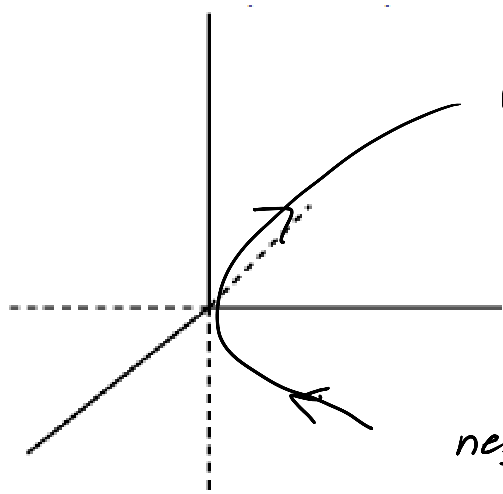
(b) $\mathbf{r}(t) = \langle \underline{t}, t^2, \underline{\underline{c}} \rangle$, where c is a constant. $(c \neq 0)$



(c) $\mathbf{r}(t) = \langle t, t^2, t \rangle$.

$\left. \begin{array}{l} x = t \\ y = t^2 \end{array} \right\} \to y = x^2$

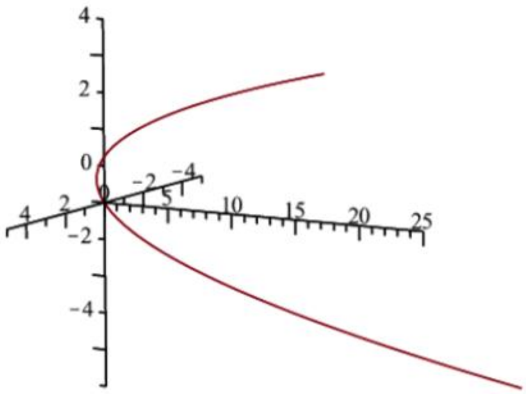
$\left. \begin{array}{l} z = t \\ x = t \end{array} \right\} \to x = z$

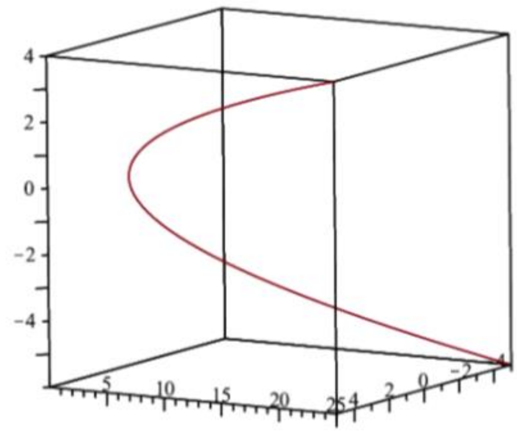


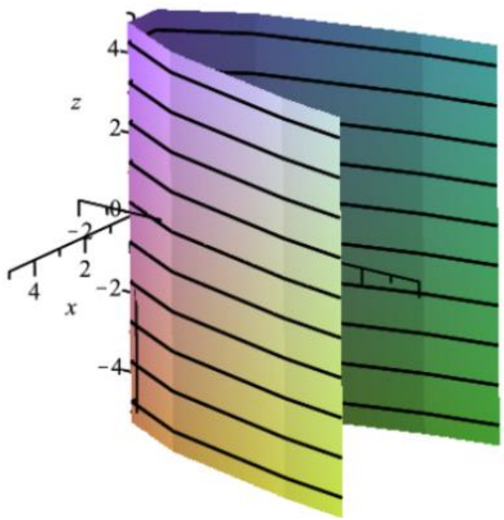
positive x + z values.

negative x + z values

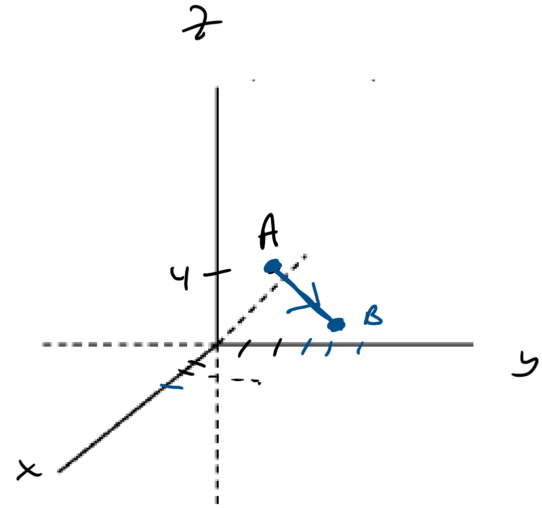
graphs







(d) $\mathbf{r}(t) = \langle 2+t, 2+3t, 4-2t \rangle$, $0 \leq t \leq 1$



$r(0) = \langle 2, 2, 4 \rangle$ A

$r(1) = \langle 3, 5, 2 \rangle$ B

Sphere) $x^2+y^2+z^2=r^2$

Example: Show that the curve $\mathbf{r}(t) = \left\langle \sin(t), 2\cos(t), \sqrt{3}\sin(t) \right\rangle$ lies on both a plane and a sphere. What does the space curve for $\mathbf{r}(t)$ look like?

$x = \sin(t)$

$y = 2\cos(t)$

$z = \sqrt{3}\sin t$

$z = \sqrt{3}\,x$ plane.

Sphere ??

$$x^2+y^2+z^2 = (\sin t)^2 + (2\cos t)^2 + (\sqrt{3}\sin t)^2$$

$$= \sin^2 t + 4\cos^2 t + 3\sin^2 t$$

$$= 4\sin^2 t + 4\cos^2 t$$

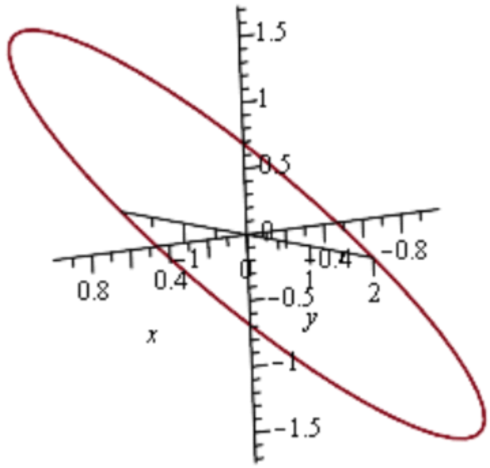
$$= 4(\sin^2 t + \cos^2 t)$$

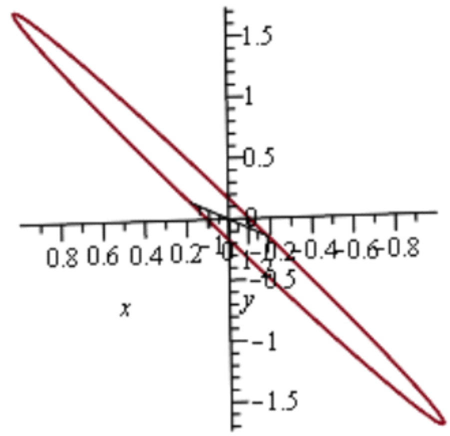
$$x^2+y^2+z^2 = 4$$

sphere centered at origin

Radius 2

Graphs 2





y-axis is comming out of the screen.

$\sin 2\theta = 2\sin\theta\cos\theta$

Example: Find a vector function that represents the curve of intersection of the two surfaces.
$x^2 + y^2 = 4$ and $z = xy$

$z = 2\cos\theta \, 2\sin\theta$

$z = 4\cos\theta\sin\theta$

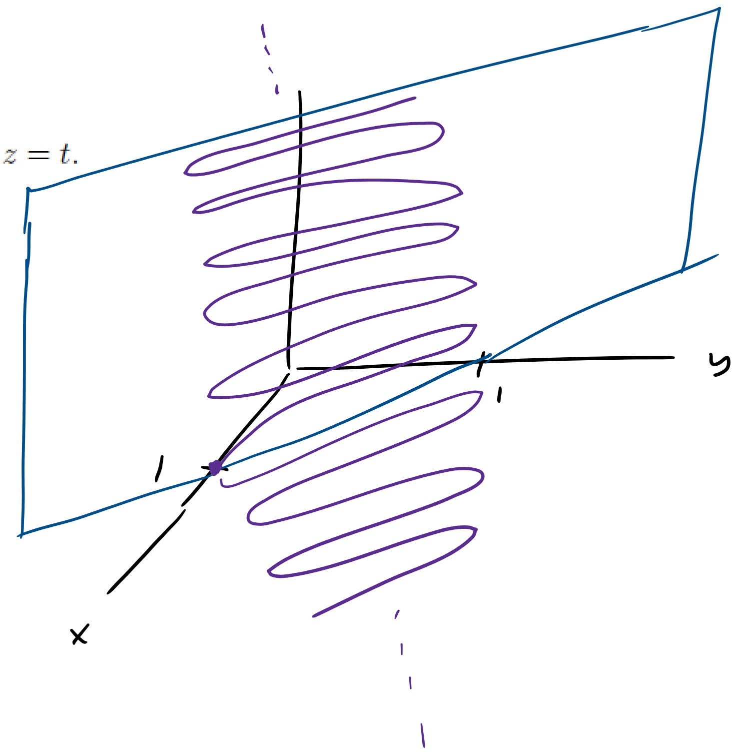
$z = 2\sin 2\theta$

$x = 2\cos\theta$

$y = 2\sin\theta$

$r(\theta) = \langle 2\cos\theta, 2\sin\theta, 4\cos\theta\sin\theta \rangle$

Example: Sketch the curve $x = \cos^2 t$, $y = \sin^2 t$, and $z = t$.



$\cos^2 t + \sin^2 t = 1$

$x + y = 1$