Section 12.4: The Cross Product

Reviewing the Determinate

The determinate of a 2x2 matrix is computed by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinate of a 3x3 matrix is computed by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \overset{+}{=} a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example: Find the determinate of this matrice.

$$\begin{vmatrix} 1 & 3 & 4 \\ 5 & 0 & 2 \\ -3 & 6 & 7 \end{vmatrix} = 1 \begin{vmatrix} 0 & 2 \\ 6 & 7 \end{vmatrix} - 3 \begin{vmatrix} 5 & 2 \\ -3 & 7 \end{vmatrix} + 4 \begin{vmatrix} 5 & 0 \\ -3 & 6 \end{vmatrix}$$

$$= 1\{0(7) - (2)(6)\} - 3\{5(7) - (-3)(2)\} + 4\{5(6) - (-3)(0)\}$$

$$= 1(0 - 12) - 3(35 - -6) + 4(30 - 0)$$

$$= 1(-12) - 3(41) + 4(30)$$

$$= -12 - 123 + 120 = -15$$

Example: Find the determinate of this matrice.

$$\begin{vmatrix} 1 & 3 & 4 \\ 5 & 0 & 2 \\ -3 & 6 & 7 \end{vmatrix}$$

$$\begin{array}{c} {}^{1} \\ 35 \\ \times\ 3 \\ \hline 105 \end{array}$$

$$\left|\begin{array}{ccc|cc} 1 & 3 & 4 & 1 & 3 \\ 5 & 0 & 2 & 5 & 0 \\ -3 & 6 & 7 & -3 & 6 \end{array}\right.$$

$$= 1(0)(7) + (3)(2)(-3) + 4(5)(6) - (-3)(0)(4) - (6)(2)(1) - (7)(5)(3)$$

$$= 0 + -18 + 120 - 0 - 12 - 105$$

$$= 120 - 30 - 105$$

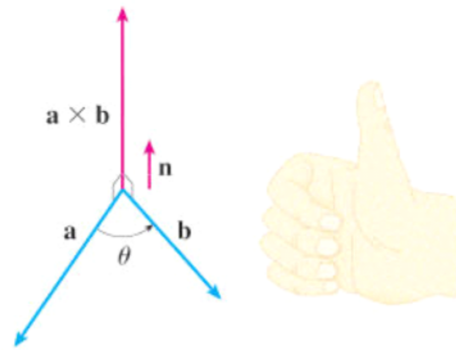
$$= -15$$

Pg 2: cross product

Definition: If $\mathbf{a}$ and $\mathbf{b}$ are two nonzero three-dimensional vectors, the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}|\sin(\theta))\,\mathbf{n}$$

where θ is the angle, $0 \leq \theta \leq \pi$, between $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{n}$ is a **unit vector** perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and whose direction is given by the **right-hand rule**: If the fingers of your right hand curls through the angle θ from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{n}$.

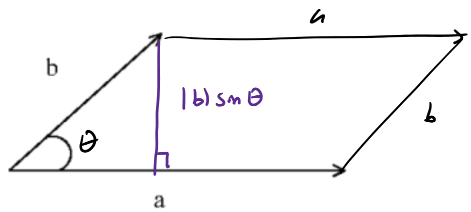


Note: $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$

Note: Two non-zero vectors, $\mathbf{a}$ and $\mathbf{b}$, are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

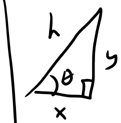
$\vec{a} \times \vec{b} = \underbrace{(|a||b|\sin\theta)}\vec{n}$

Geometric Interpretation:



$|a \times b| = \big|\, |a||b|\sin\theta \,\big|$

is Area of the parallelogram created by the vectors a + b.



$\sin\theta = \frac{y}{h}$

$h\sin\theta = y$

Properties of the Cross Product: If **a**, **b**, and **c** are vectors and d is a scalar, then

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}) = -\mathbf{b} \times \mathbf{a}$
- $(d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (d\mathbf{b})$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

Definition: If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2,\ \ a_3b_1 - a_1b_3,\ \ a_1b_2 - a_2b_1 \rangle$$

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

$$= (a_2b_3 - b_2a_3)i - (a_1b_3 - b_1a_3)j + (a_1b_2 - b_1a_2)k$$

Example: Compute the following for the vectors $\mathbf{a} = \langle 1, 3, 4\rangle$ and $\mathbf{b} = \langle 2, -5, 6\rangle$.

A) $\mathbf{a} \times \mathbf{b}$ $= \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & -5 & 6 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ -5 & 6 \end{vmatrix} i - \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix} j + \begin{vmatrix} 1 & 3 \\ 2 & -5 \end{vmatrix} k$

$$= \left[(3)(6) - (-5)(4)\right] i - \left[(1)(6) - (2)(4)\right] j + \left[1(-5) - (2)(3)\right] k$$

$$= (18 - {}^{-}20)\, i - (6 - 8)\, j + (-5 - 6)\, k$$

$$= 38i + 2j - 11k = \langle 38, 2, -11\rangle$$

$$\left|\begin{array}{ccc|cc} i & j & k & i & j \\ 1 & 3 & 4 & 1 & 3 \\ 2 & -5 & 6 & 2 & -5 \end{array}\right.$$

$$18i + 8j - 5k - 6k - {}^{-}20i - 6j$$

$$= 38i + 2j - 11k$$

B) $\mathbf{b} \times \mathbf{a}$ $= -(a \times b) = \langle -38, -2, 11\rangle$

C) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ $= 0$

Example: Find a vector orthogonal to the plane determined by the points $A(1, 2, 3)$, $B(4, 6, 8)$, and $C(15, 2, -5)$

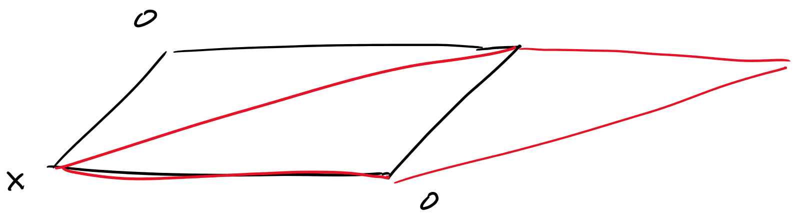
$\overrightarrow{AB} = \langle 3, 4, 5 \rangle$

$\overrightarrow{AC} = \langle 14, 0, -8 \rangle$

$\overrightarrow{AB} \times \overrightarrow{AC} = \dots = \langle -32, 94, -56 \rangle$

Example: Find the area of the parallelogram with vertices: $P(1,1,2)$, $Q(6,1,2)$, $R(4,5,5)$, and $S(9,5,5)$

$\|a \times b\|$ = area of the parallelogram created by vectors a + b



$\vec{PQ} = \langle 5, 0, 0 \rangle$

$\vec{PR} = \langle 3, 4, 3 \rangle$

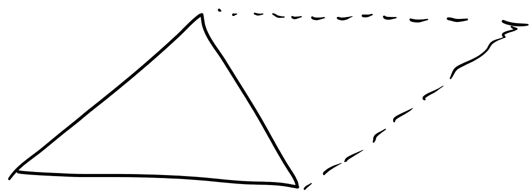
$\vec{PQ} \times \vec{PR} = \cdots = \langle 0, -15, 20 \rangle$

$$\text{Area} = |\vec{PQ} \times \vec{PR}| = \sqrt{0^2 + (-15)^2 + 20^2}$$

$$= \sqrt{0 + 225 + 400}$$

$$= \sqrt{625} = 25$$

Example: Find the area of the triangle determined by the points $P(1,1,2)$, $Q(6,1,2)$, and $R(4,5,5)$.



from the last problem

$$\text{Area} = \frac{1}{2}\left|\vec{PQ} \times \vec{PR}\right|$$

$$= \frac{1}{2}(25) = 12.5$$

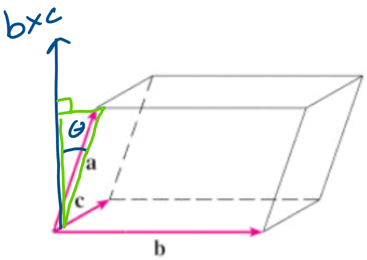
Definition: If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ are vectors, then the **scalar triple product** is given by

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k} \right)$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note: The geometric interpretation of scalar triple product is that its magnitude is the volume of the parallelepiped formed by the vectors: **a**, **b**, and **c**.



$a \cdot (b \times c) = |a|\,|b \times c| \cos\theta$

$|b \times c| =$ area of parallelogram created by vectors $b + c$

$|a| \cos\theta =$ "height" of the object.

$|a \cdot (b \times c)| =$ Volume of the parallelpiped created by the 3 vectors.

Example: Compute a scalar triple product of these vectors: $\mathbf{a} = \langle 1, 2, 3\rangle$, $\mathbf{b} = \langle 4, 5, 6\rangle$, and $\mathbf{c} = \langle 2, 7, 5\rangle$ Are these vectors co-planer?

$$\left|\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 7 & 5 \end{matrix}\right| \begin{matrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{matrix}$$

$$a \cdot (b \times c) = \left|\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 7 & 5 \end{matrix}\right|$$

$$\begin{array}{r} {}^{1}12 \\ 7 \\ \hline 84 \end{array}$$

$$= 25 + 24 + 84 - 30 - 42 - 40$$

$$= 21$$

not co-planer

Example: Determine if these points are co-planer: $A(4, -3, 1)$, $B(6, -4, 7)$, $C(1, 2, 2)$, and $D(0, 1, 11)$

$\overrightarrow{AB} = \langle 2, -1, 6 \rangle$

$\overrightarrow{AC} = \langle -3, 5, 1 \rangle$

$\overrightarrow{AD} = \langle -4, 4, 10 \rangle$

$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \ldots\ldots = 114$

Not co-planer