# Section 12.3: The Dot Product

**Definition:** The **dot product** of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is the number

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where $\theta$ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, $0 \le \theta \le \pi$. If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, then $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$.

The **dot product** of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

**Definition:** Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal(perpendicular) if and only if $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$, i.e. the angle between them is $\pi/2$.

Example: Find the following using these vectors: $\mathbf{a} = \langle -1, -2, -3 \rangle$, $\mathbf{b} = \langle -10, 2, 1 \rangle$, and $\mathbf{c} = \langle 2, 8, -6 \rangle$.

A) $\mathbf{a} \cdot \mathbf{b} =$ $(-1)(-10) + (-2)(2) + (-3)(1)$

$= 10 - 4 - 3 = 10 - 7 = 3$

B) $\mathbf{a} \cdot \mathbf{c} =$ $(-1)(2) + (-2)(8) + (-3)(-6)$

$= -2 - 16 + 18$

$= 0$

vectors a & c are perp.

C) Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

$a = \langle -1, -2, -3 \rangle$

$b = \langle -10, 2, 1 \rangle$

$a \cdot b = |a|\,|b| \cos\theta$

$3 = \sqrt{14}\sqrt{105} \cos\theta$

$\cos\theta = \boxed{\dfrac{3}{\sqrt{14}\sqrt{105}}}$

$\boxed{\theta = \arccos \dfrac{3}{\sqrt{14}\sqrt{105}}} = 85.51^\circ = 1.4925$ Radians.

$|a| = \sqrt{(-1)^2 + (-2)^2 + (-3)^2}$

$= \sqrt{1 + 4 + 9} = \sqrt{14}$

$|b| = \sqrt{(-10)^2 + 2^2 + 1^2}$

$= \sqrt{100 + 4 + 1} = \sqrt{105}$

Example: If $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 2$, what is the maximum for $\mathbf{a} \cdot \mathbf{b}$? What does this say about the vectors?

$$a \cdot b = |a|\,|b| \cos\theta$$
$$= (1)(2) \cos\theta$$
$$= 2\cos\theta$$

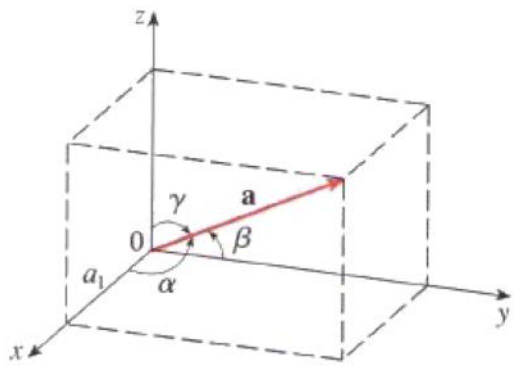
max of $a \cdot b = 2$ when $\cos\theta = 1$

$\theta = 0$

min of $a \cdot b = -2$ When $\cos\theta = -1$

$\theta = \pi$

**Directional angles/and Direction Cosines**




find Angle between $a = \langle a_1, a_2, a_3 \rangle$ & x-axis.

$i = \langle 1, 0, 0 \rangle$

$a \cdot i = |a|\,|i| \cos\alpha$

$a_1 = |a| \cos\alpha$

$\cos\alpha = \frac{a_1}{|a|}$ ie $\alpha = \arccos \frac{a_1}{|a|}$

y-axis $\beta$

$\cos\beta = \frac{a_2}{|a|}$ or $\beta = \arccos \frac{a_2}{|a|}$

z-axis $\gamma$

$\cos\gamma = \frac{a_3}{|a|}$ or $\gamma = \arccos \frac{a_3}{|a|}$

Example: Find the direction angles for $\mathbf{a} = \langle 1, 0, 5 \rangle$

$|a| = \sqrt{1^2 + 0^2 + 5^2} = \sqrt{26}$

x-axis

$\alpha = \arccos \frac{1}{\sqrt{26}}$

y-axis

$\beta = \arccos\left(\frac{0}{\sqrt{26}}\right) = \frac{\pi}{2}$

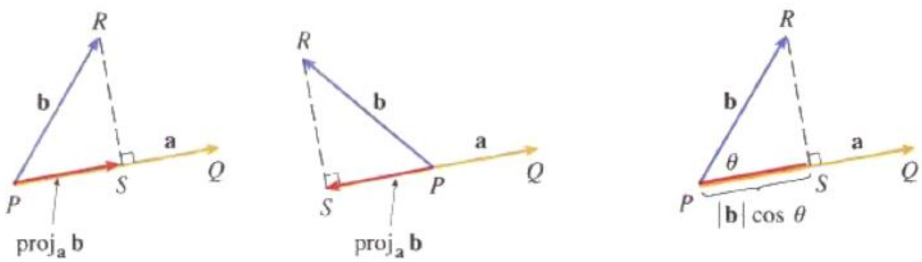
z-axis

$\gamma = \arccos \frac{5}{\sqrt{26}}$

**Projections**

Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: $\quad \text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of $\mathbf{b}$ onto $\mathbf{a}$: $\quad \text{proj}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$




Vector projections

Scalar projection

Example: Find the vector and scalar projections of $\mathbf{m} = \langle 2, 1, 5\rangle$ onto $\mathbf{n} = \langle 1, 2, 3\rangle$

$|n| = \sqrt{1^2+2^2+3^2} = \sqrt{1+4+9}$
$= \sqrt{14}$

Scalar. $\quad \text{Comp}_n m = \dfrac{m \cdot n}{|n|} = \dfrac{2+2+15}{\sqrt{14}} = \dfrac{19}{\sqrt{14}}$

Vector proj $\quad \text{proj}_n m = \dfrac{m \cdot n}{|n|^2}\, n = \dfrac{19}{(\sqrt{14})^2}\langle 1,2,3\rangle = \dfrac{19}{14}\langle 1,2,3\rangle$