Solutions and questions can be found at the link:
https://www.math.tamu.edu/~kahlig/152WIR.html

1. Use the Comparison Theorem to determine if the integral converges or diverges

$$\int_2^\infty \frac{4x}{x^2+e^{4x^2}}\,dx$$

$x^2 + e^{4x^2} > x^2$

$\frac{1}{x^2+e^{4x^2}} < \frac{1}{x^2}$

$\frac{4x}{x^2+e^{4x^2}} < \frac{4x}{x^2} = \frac{4}{x}$

$\int_2^\infty \frac{4}{x}\,dx$ p-integral $p=1$ div.

no use for comparison thrm

$x^2 + e^{4x^2} > e^{4x^2}$

$\frac{1}{x^2+e^{4x^2}} < \frac{1}{e^{4x^2}}$

$\frac{4x}{x^2+e^{4x^2}} < \frac{4x}{e^{4x^2}} = 4xe^{-4x^2}$

need to look at

$\int_2^\infty 4xe^{-4x^2}\,dx$

$$\int_2^\infty 4xe^{-4x^2}\,dx = \lim_{t\to\infty}\int_2^t 4xe^{-4x^2}\,dx = \lim_{t\to\infty} -\frac{1}{2}e^{-4x^2}\Big|_2^t$$

$$= \lim -\frac{1}{2}e^{-4t^2} - -\frac{1}{2}e^{-16}$$

$$\boxed{\begin{aligned} \int 4x e^{-4x^2} dx &= \int \frac{-4}{8} e^u du \\ u = -4x^2 \quad &= \frac{-1}{2} e^u \\ du = -8x\,dx \quad &= \frac{-1}{2} e^{-4x^2} \\ -\frac{1}{8} du = x\,dx \quad & \end{aligned}}$$

$$= \lim_{t\to\infty} \frac{-1}{2} e^{-4t^2} - \frac{-1}{2} e$$

$$= \lim_{t\to\infty} \left(\underbrace{\frac{-1}{2e^{4t^2}}} + \frac{1}{2e^{16}} \right)$$

$$= 0 + \frac{1}{2e^{16}}$$

$$= \frac{1}{2e^{16}}$$

Thus $\int_2^\infty 4x e^{-4x^2} dx$ conv.

Thus by comparison $\int_2^\infty \frac{4x}{x^2+e^{4x^2}} dx$ will conv.

2. Find a formula for the general term, a_n, of the sequence assuming that the pattern of the first few terms continues. Give the formula so the first term is a_1.

$$\left\{\frac{1}{5}, \frac{-4}{9}, \frac{7}{13}, \frac{-10}{17}, \frac{13}{21}, \dots\right\}$$

a_1

Top has inc. 3

$m = 3$

n

$\cancel{x}, y$

$(1, 1)$

$(2, 4)$

$m = \frac{4-1}{2-1} = 3$

$y - 1 = 3(x-1)$

$y - 1 = 3x - 3$

$y = 3x - 2$

$$\boxed{a_n = \frac{(-1)^{n+1}(3n-2)}{4n+1}}$$

$n = 1, 2, 3, 4, \dots$

Bottom inc. of 4

$m = 4$

$(1, 5)$ — x, y

$y - 5 = 4(x-1)$

$y = 4x - 4 + 5$

$y = 4x + 1$

If needed you can shift a series

start at $j = 5$ (ie $n = 5$)

$j = n + 4 \rightarrow j - 4 = n$

$$a_j = \frac{(-1)^{j-3}(3(j-4)-2)}{4(j-4)+1}$$

3. Determine whether each sequence converges or diverges. If it converges, find the limit.

(a) $a_n = \frac{3n}{\ln(4n)}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n}{\ln(4n)} \overset{L'H}{=} \lim_{n\to\infty} \frac{3}{\frac{4}{4n}} = \lim_{n\to\infty} 3 \cdot \frac{4n}{4} = \infty$$

The seq. diverges

(b) $a_n = \frac{(-3)^n}{2^{2n}}$ $= (-1)^n \frac{3^n}{2^{2n}}$

$$b_n = \frac{3^n}{2^{2n}} = \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$$

as $n \to \infty$ $b_n \to 0$

Thus a_n converges to zero.

If $a_n = (-1)^n b_n$

and b_n converges to zero.

Then a_n converges to zero.

otherwise a_n diverges

(c) $a_n = \frac{(-1)^n n^2}{n^2+1}$ $= (-1)^n \frac{n^2}{n^2+1}$

$$b_n = \frac{n^2}{n^2+1}$$

as $n \to \infty$ $b_n \to 1$

Thus a_n diverge.

(d) $a_n = \sin(e^{-2n})$

$$\lim_{n\to\infty} \sin\left(e^{-2n}\right) = \sin(0) = 0$$

seq. converges to 0

(e) $a_n = \dfrac{\sin(3n)}{n+5}$

seq. converges to zero.

$$-1 \le \sin(3n) \le 1$$

$$b_n = \frac{-1}{n+5} \le \frac{\sin(3n)}{n+5} \le \frac{1}{n+5} = c_n$$

As $n \to \infty$

$b_n \to 0$ & $c_n \to 0$

by squeeze thm

seq. converges to 0.

4. Determine whether the following sequence are increasing, decreasing, or not monotonic. Also determine if the sequence is bounded.

(a) $a_n = 3 + \dfrac{1}{n}$

$f(x) = 3 + \dfrac{1}{x}$

$f' = -\dfrac{1}{x^2}$

for $x > 0$ $f' < 0$

for $x > 0$ dec.

bounded → need to find Abs max & Abs min for the seq.

OR If seq. converges then we know it is bounded

$\lim_{n \to \infty} a_n = 3$

The seq. is convergent so It has to be bounded.

(b) $a_n = \ln(2+3n) - \ln(1+n)$

$$\lim_{n\to\infty} \ln\left(\frac{2+3n}{1+n}\right) = \ln\left(\frac{3}{1}\right)$$
$$= \ln(3)$$

Seq. is conv.
Thus it is bounded

$$f(x) = \ln(2+3x) - \ln(1+x)$$

$$f'(x) = \frac{3}{2+3x} - \frac{1}{x+1} = \frac{3(x+1) - 1(2+3x)}{(2+3x)(x+1)}$$

$$= \frac{3x+3-2-3x}{(2+3x)(x+1)} = \frac{1}{(2+3x)(x+1)} = f'(x)$$

$f'(x)$ not defined $x = -\frac{2}{3}$ & $x = -1$

for $x > 0$ $f'(x) > 0$ $f(x)$ Inc.

Seq. is inc.

(c) $a_n = \cos\left(\frac{1}{n}\right)$

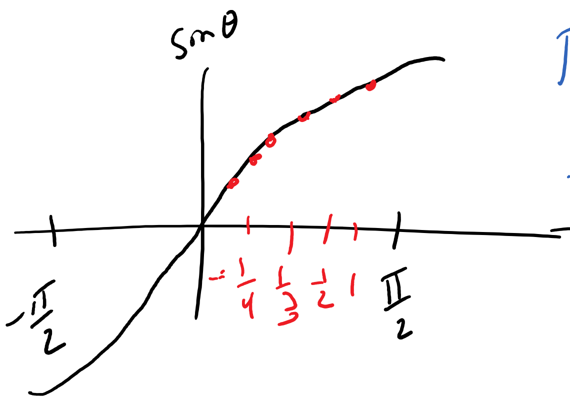
yes it is bounded

$$-1 \le \cos\left(\frac{1}{n}\right) \le 1$$

$$f(x) = \cos\left(\frac{1}{x}\right)$$

$$f'(x) = -\sin\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2} = \frac{\sin\left(\frac{1}{x}\right)}{x^2} > 0$$

for Integer values of $x > 0$ $\sin\left(\frac{1}{x}\right)$ is positive.



Thus $f(x)$ Is Inc

if $x=1$ $\sin(1)$

$x=2$ $\sin\left(\frac{1}{2}\right)$

$x=3$ $\sin\left(\frac{1}{3}\right)$

5. Assume that the sequence defined below is bounded and is decreasing. Determine if the sequence is convergent or divergent.

$$a_1 = 2, \qquad a_{n+1} = \frac{6}{7 - a_n}$$

seq. conv.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{6}{7-a_n}$$

$$L = \frac{6}{7-L}$$

$$L(7-L) = 6$$

$$7L - L^2 = 6$$

$$0 = L^2 - 7L + 6$$

$$(L-6)(L-1)$$

$L = 6$

$L = 1$

Since $a_1 = 2$ + a_n dec.

Seq conv. to $L = 1$

6. Let $a_n = \dfrac{2n^2}{5n^2-3}$

(a) Determine whether $\{a_n\}$ is convergent.

$\lim_{n\to\infty} a_n = \frac{2}{5}$

seq.

Seq. Converges.

(b) Determine whether $\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{Series.}}$ is convergent

div.

Test of divergence for $\sum a_n$ if $\lim_{n\to\infty} a_n \neq 0$ Then series will diverge

7. Assume the n-th partial sum of the series $\sum_{n=1}^{\infty} a_n$ is $s_n = \dfrac{3n+2}{1-2n}$.

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

$S_4 = S_3 + a_4$

$S_4 - S_3 = a_4$

(a) Find a_1 and a_4.

$$a_1 = S_1 = \frac{3+2}{1-2} = \frac{5}{-1} = -5$$

$$a_4 = S_4 - S_3 = \frac{14}{-7} - \frac{11}{-5}$$

(b) Find a formula for a_n when $n > 1$

$$a_n = S_n - S_{n-1}$$

(c) Find $\sum_{n=1}^{\infty} a_n$. $= \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3n+2}{1-2n} = \frac{3}{-2} = \frac{-3}{2} = -1.5$

8. Determine whether the following series are convergent or divergent. If a series is convergent, find its sum.

(a) $\sum_{n=1}^{\infty} \ln\left(\frac{3e^{2n}}{e^{2n}+4}\right)$

Test for div.

$$\lim_{n\to\infty} \ln\left(\frac{3e^{2n}}{e^{2n}+4}\right) = \ln(3) \neq 0$$

series will diverge by the Test for div.

$$\lim_{n\to\infty} \frac{3e^{2n}}{e^{2n}+4} \overset{L'H}{=} \lim_{n\to\infty} \frac{6e^{2n}}{2e^{2n}} = \lim_{n\to\infty} \frac{6}{2} = 3$$

(b) $\sum_{n=2}^{\infty} 5^{-n+1} 3^n$

$= 5^{-1}3^2 + 5^{-2}3^3 + 5^{-3}3^4 + 5^{-4}3^5 + \cdots$

$n=2 \qquad n=3 \qquad n=4 \qquad n=5$

$= \frac{3^2}{5} + \frac{3^3}{5^2} + \frac{3^4}{5^3} + \frac{3^5}{5^4} + \cdots$

$a \qquad ar \qquad ar^2$

geometric series.

$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0} ar^n$

$= a + ar + ar^2 + ar^3 + \cdots\cdots$

if $|r|<1$ then it conv.

and $\text{sum} = \frac{a}{1-r}$

$r = \frac{3}{5}$ $\qquad a = \frac{3^2}{5} = \frac{9}{5}$

Is $|r|<1$? yes.

The series converges.

$\text{Sum} = \frac{a}{1-r} = \frac{\frac{9}{5}}{1-\frac{3}{5}} = \frac{\frac{9}{5}}{\frac{2}{5}} = \frac{9}{2}$

If you need help to find r

$\frac{ar}{a} = r$

(c) $8 - 10 + \frac{25}{2} - \frac{125}{8} + \frac{625}{32} - \cdots$

a ar

$\frac{ar}{a} = \frac{-10}{8} = -\frac{5}{4} = r$

$+\frac{50}{4} = +\frac{25}{2}$ ✓

$-\frac{125}{8}$ ✓

Is $|r|<1$?

no.

The series diverges

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n + 3^n}{5^n}$

$n=1$ $\quad \frac{-1+3}{5} = \frac{2}{5}$

$n=2$ $\quad \frac{1+9}{25} = \frac{10}{25}$

$n=3$ $\quad \frac{-1+27}{125} = \frac{26}{125}$

not geometric

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n} + \sum_{n=1}^{\infty} \frac{3^n}{5^n}$$

$$= \frac{-1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \frac{1}{5^4} + \cdots + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right)^3 + \cdots$$

$a = \frac{-1}{5} \quad r = \frac{-1}{5}$ $\qquad$ $a = \frac{3}{5} \quad r = \frac{3}{5}$

$$\text{Sum} = \frac{-\frac{1}{5}}{1 - \frac{-1}{5}} + \frac{\frac{3}{5}}{1 - 3/5}$$

$$= \frac{-\frac{1}{5}}{6/5} + \frac{3/5}{2/5}$$

Converges

$$= -\frac{1}{6} + \frac{3}{2} = \frac{-1}{6} + \frac{9}{6} = \frac{8}{6}$$

9. Find the values of x for which the following series converge. Give your answer in interval notation. Find the sum of the series.

$$\sum_{n=1}^{\infty} 2^{n-1}(x-3)^n = 2^0(x-3)^1 + 2(x-3)^2 + 2^2(x-3)^3 + 2^3(x-3)^4 + \cdots$$

$n=1$ a $\quad$ $n=2$ ar $\quad$ $n=3$ ar^2 $\quad$ $n=4$

$a = 2^0(x-3)$ $\qquad$ $r = 2(x-3)$

$a = x-3$

$$r = \frac{ar}{a} = \frac{2(x-3)^2}{(x-3)} = 2(x-3)$$

Converge when

$$|r| < 1$$

$$|2(x-3)| < 1$$

$$|2x-6| < 1$$

$$-1 < 2x-6 < 1$$

$+6$ $\quad$ $+6$ $\quad$ $+6$

$$5 < 2x < 7$$

$$\frac{5}{2} < x < \frac{7}{2}$$

Interval of convergence.

Interval notation

$$\left(\frac{5}{2}, \frac{7}{2}\right)$$

$$\text{Sum} = \frac{a}{1-r} = \frac{x-3}{1-2(x-3)} = \frac{x-3}{1-2x+6} = \frac{x-3}{7-2x}$$

10. Determine if this telescoping series converges or diverges. If it converges give the value.

$$\sum_{k=1}^{\infty}\left(e^{3/(k+2)} - e^{3/k}\right)$$

→ need to find a formula for S_n

$$S_n = e^{\frac{3}{3}} - e^{3/1} \quad K=1$$
$$+\ e^{\frac{3}{4}} - e^{3/2} \quad K=2$$
$$+\ e^{\frac{3}{5}} - e^{3/3} \quad K=3$$
$$+\ e^{\frac{3}{6}} - e^{3/4} \quad K=4$$
$$+\ e^{3/7} - e^{3/5} \quad K=5$$
$$\vdots$$

$$\vdots$$
$$+\ e^{\frac{3}{n-1}} - e^{\frac{3}{n-3}} \quad K=n-3$$
$$+\ e^{\frac{3}{n}} - e^{\frac{3}{n-2}} \quad K=n-2$$
$$+\ e^{\frac{3}{n+1}} - e^{\frac{3}{n-1}} \quad K=n-1$$
$$+\ e^{\frac{3}{n+2}} - e^{\frac{3}{n}} \quad K=n$$

$$S_n = \underbrace{-e^{3/1} - e^{3/2} + e^{\frac{3}{n+1}} + e^{\frac{3}{n+2}}}$$

The Series will conver if $\lim\limits_{n\to\infty} S_n$ exists.

$$\text{Sum} = \lim_{n\to\infty} S_n = -e^{3} - e^{3/2} + e^{0} + e^{0}$$
$$= -e^{3} - e^{3/2} + 1 + 1$$
$$= \underbrace{2 - e^{3} - e^{3/2}}$$

← Series Converges to this #