Section 7.8: Improper Integrals

$\int_a^b f(x)\,dx$

Improper integrals of Type I

(a) If $\int_a^t f(x)\,dx$ exist for every number $t \geq a$ then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$ provided this limit exists (as a finite number).

(b) If $\int_t^a f(x)\,dx$ exist for every number $t \leq a$ then $\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx$ provided this limit exists (as a finite number).

The improper integrals in (a) and (b) are called convergent if the limits exists and divergent if the limit does not exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define $\int_{-\infty}^\infty f(x)\,dx = \int_a^\infty f(x)\,dx + \int_{-\infty}^a f(x)\,dx$

Compute these integrals.

A) $\int_2^\infty \frac{1}{(x+3)^{1.5}}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{(x+3)^{1.5}}\,dx = \lim_{t\to\infty} \left.\frac{-2}{\sqrt{x+3}}\right|_2^t$

Side work

$$\int \frac{1}{(x+3)^{1.5}}\,dx = \int \frac{1}{u^{1.5}}\,du$$

$u = x+3$

$du = dx$

$$= \int u^{-1.5}\,du$$

$$= \frac{u^{-.5}}{-\frac{1}{2}} + C$$

$$= \frac{-2}{\sqrt{x+3}}$$

$$= \lim_{t\to\infty} \frac{-2}{\sqrt{t+3}} + \frac{+2}{\sqrt{5}}$$

$$= 0 + \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$

We know the will converge and it converges to the value of $\frac{2}{\sqrt{5}}$

B) $\int_0^\infty \frac{\cos(x)}{1+\sin^2(x)}\,dx = \lim_{t\to\infty} \int_0^t \frac{\cos(x)}{1+\sin^2(x)}\,dx = \lim_{t\to\infty} \arctan(\sin(x))\Big|_0^t$

$\int \frac{\cos(x)}{1+\sin^2(x)}\,dx = \int \frac{1}{1+u^2}\,du$

$u = \sin(x)$

$du = \cos(x)\,dx$

$= \arctan(u)$

$= \arctan(\sin(x))$

$= \lim_{t\to\infty} \arctan(\sin(t)) - \arctan(\sin(0))$

$=$ DNE

The Integral will diverge

C) $\displaystyle\int_1^\infty \frac{x+1}{(x+3)(x+4)}\,dx$

$$\int \frac{x+1}{(x+3)(x+4)}\,dx = \int \frac{-2}{x+3} + \frac{3}{x+4}\,dx$$

partial fractions

$$= -2\ln|x+3| + 3\ln|x+4|$$

C) $\displaystyle\int_1^\infty \frac{x+1}{(x+3)(x+4)}\,dx = \lim_{t\to\infty} -2\ln(x+3) + 3\ln(x+4)\Big|_1^t$

$$-\infty \quad + \infty$$

$$= \lim_{t\to\infty} -2\ln(t+3) + 3\ln(t+4) - \left[-2\ln(4) + 3\ln(5)\right]$$

$$= \infty$$

This integral diverges

$$\lim_{t\to\infty} 3\ln(t+4) - 2\ln(t+3) = \lim_{t\to\infty} \ln(t+4)^3 - \ln(t+3)^2$$

$$= \lim_{t\to\infty} \ln\left(\frac{(t+4)^3}{(t+3)^2}\right) = \infty$$

$$\lim_{t\to\infty} \frac{(t+4)^3}{(t+3)^2} = \lim_{t\to\infty} \frac{t^3 + \cdots\cdots + 4^3}{t^2+6t+9} = \infty$$

Fact: The $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and diverges if $p \le 1$.

Example: For what values of p will these integrals converge?

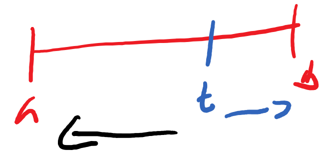
$\int_1^\infty \frac{5}{x^p}\,dx$ $\qquad$ $\int_A^\infty \frac{5}{x^p}\,dx$, where $A > 0$.

$$5\int_1^\infty \frac{1}{x^p}\,dx$$

$$\int_1^\infty \frac{5}{x^p}\,dx = \boxed{\int_1^A \frac{5}{x^p}\,dx} + \int_A^\infty \frac{5}{x^p}\,dx$$

Page 6: Type II

Improper integrals of Type II



(a) If f is continuous on $[a, b)$ and is discontinuous at b, then $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$ provided this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$ provided this limit exists (as a finite number).

The improper integrals in (a) and (b) are called convergent if the limits exists and divergent if the limit does not exist.

(c) If f has a discontinuity at c where $a < c < b$, and both $\displaystyle\int_a^c f(x)\,dx$ and $\displaystyle\int_c^b f(x)\,dx$ are convergent, then we define $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Compute these integrals.

A) $\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{t \to 0^+} \int_t^8 x^{-1/3}\,dx = \lim_{t \to 0^+} \frac{3}{2}x^{2/3}\Big|_t^8$

$= \lim_{t \to 0^+} \frac{3}{2}(8)^{2/3} - \frac{3}{2}t^{2/3}$

$= \frac{3}{2}(8)^{2/3} = \frac{3}{2}(4) = 6$

This integral converges to 6.

diverges

$$\text{B)} \int_{-3}^{1} \frac{1}{x^2}\,dx = \int_{-3}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

$$\underbrace{\int_{0}^{1} \frac{1}{x^2}\,dx}_{\text{diverges}} = \lim_{t\to 0^+} \int_{t}^{1} x^{-2}\,dx = \lim_{t\to 0^+} \left.\frac{x^{-1}}{-1}\right|_{t}^{1}$$

$$= \lim_{t\to 0^+} \left.-\frac{1}{x}\right|_{t}^{1} = \lim_{t\to 0^+} -1 + \frac{+1}{t}$$

$$= +\infty$$

It diverges to $+\infty$.

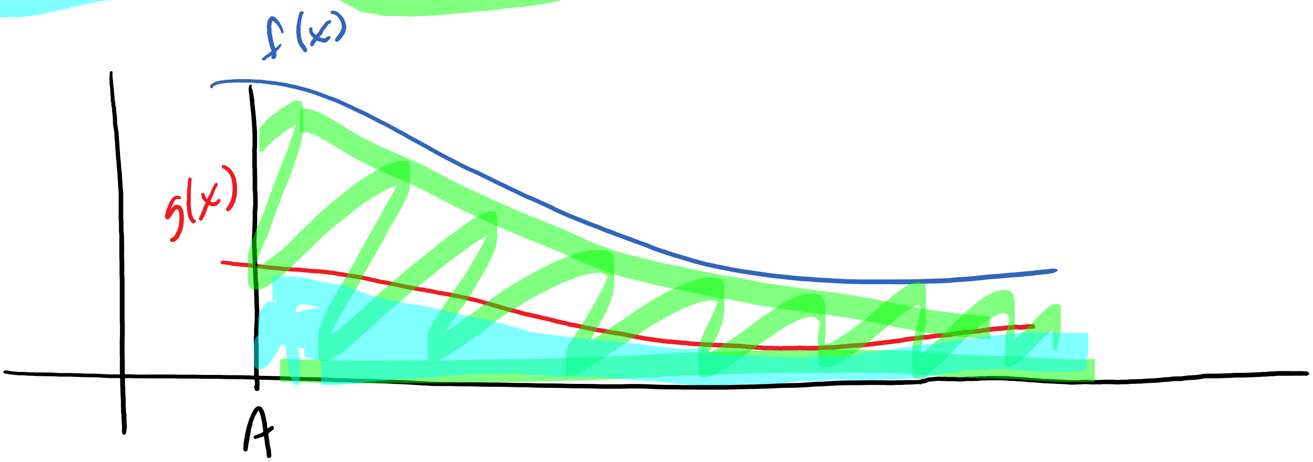
Fact: The $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ is convergent if $p < 1$ and diverges if $p \geq 1$.

Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$

(a) If $\displaystyle\int_a^\infty f(x)dx$ is convergent, then $\displaystyle\int_a^\infty g(x)dx$ is convergent.

(b) If $\displaystyle\int_a^\infty g(x)dx$ is divergent, then $\displaystyle\int_a^\infty f(x)dx$ is divergent.



Use the comparison theorem to decide if these integrals converge or diverge.

A) $\displaystyle\int_2^\infty \boxed{\frac{1}{\sqrt[3]{x^2-1}}}\,dx$

$A < B \longrightarrow A < B$

$\frac{1}{A} > \frac{1}{B}$

$\frac{A}{A} < \frac{B}{A}$

$1 < \frac{B}{A}$

$\frac{1}{B} < \frac{1}{A}$

$x^2 - 1 < x^2$

$\sqrt[3]{x^2-1} < \sqrt[3]{x^2}$

$\frac{1}{\sqrt[3]{x^2-1}} > \frac{1}{\sqrt[3]{x^2}}$

$f(x) = \frac{1}{\sqrt[3]{x^2-1}} > \frac{1}{\sqrt[3]{x^2}} = g(x)$

$$\int_2^\infty \frac{1}{\sqrt[3]{x^2}}\,dx = \int_2^\infty \frac{1}{x^{2/3}}\,dx$$

p integral
$p = 2/3$
diverges.

Thus By the comparison Thrm $\displaystyle\int_2^\infty \frac{1}{\sqrt[3]{x^2-1}}\,dx$ will diverge.

B) $\displaystyle\int_2^{\infty} \frac{1}{\sqrt[3]{x^4-1}}\,dx$

$$\int_2^{\infty} \frac{1}{\sqrt[3]{x^4}}\,dx$$

p-integral
$p = 4/3 > 1$
converges.

$$x^4 - 1 < x^4$$

$$\sqrt[3]{x^4-1} < \sqrt[3]{x^4}$$

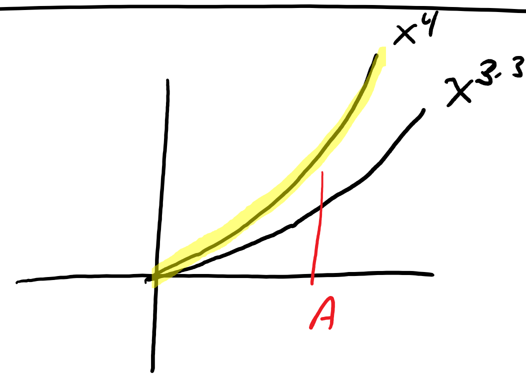
$$\underbrace{f(x) = \frac{1}{\sqrt[3]{x^4-1}} > \frac{1}{\sqrt[3]{x^4}}}_{} = g(x)$$

This comparison does not allow for a conclusion

$$x^4 - 1 > x^{3.3}$$

for $x > A$

$$\sqrt[3]{x^4-1} > \sqrt[3]{x^{3.3}} = x^{\frac{3.3}{3}} = x^{1.1}$$



$$f(x) = \frac{1}{\sqrt[3]{x^4-1}} < \frac{1}{\underbrace{x^{1.1}}} = g(x)$$

$$\int_A^{\infty} \frac{1}{x^{1.1}}\,dx$$

p-integral
$p = 1.1$
conv.

by comparison $\displaystyle\int_A^{\infty} f(x)\,dx$ will conv.

Thus $\displaystyle\int_2^{\infty} f(x)\,dx$ will conv.

C) $\int_2^\infty e^{-x^4}\,dx$

$$x^4 > x$$

$$-x^4 < -x$$

$$e^{-x^4} < e^{-x}$$

$$\int_2^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_2^t e^{-x}\,dx$$

$$= \lim_{t\to\infty} -e^{-x}\Big|_2^t$$

$$= \lim_{t\to\infty} -e^{-t} + e^{-2}$$

$$= 0 + e^{-2} = e^{-2}$$

This Integral converges.

By comparison Thm

$\int_2^\infty e^{-x^4}\,dx$ will also converge.

D) $\displaystyle\int_1^\infty \frac{3+2\cos(2x)}{x^2}\,dx = J$

$$\int_1^\infty \frac{1}{x^2}\,dx \quad \& \quad \int_1^\infty \frac{5}{x^2}\,dx$$

p-integral $p = 2$

conv.

$$-1 \le \cos(2x) \le 1$$

$$-2 \le 2\cos(2x) \le 2$$

$$1 \le 3+2\cos(2x) \le 5$$

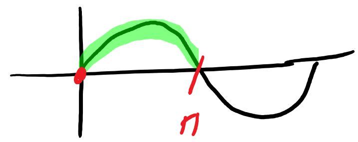
$$\frac{1}{x^2} \le \frac{3+2\cos(2x)}{x^2} \le \frac{5}{x^2}$$

Since $\int_1^\infty \frac{5}{x^2}\,dx$ conv.

we know by comp. thrm

that J will conv.

E) $\displaystyle\int_0^{\pi} \frac{\sin(x)}{\sqrt{x}}\, dx$



$0 \le \sin(x) \le 1$

$0 \le \dfrac{\sin(x)}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$

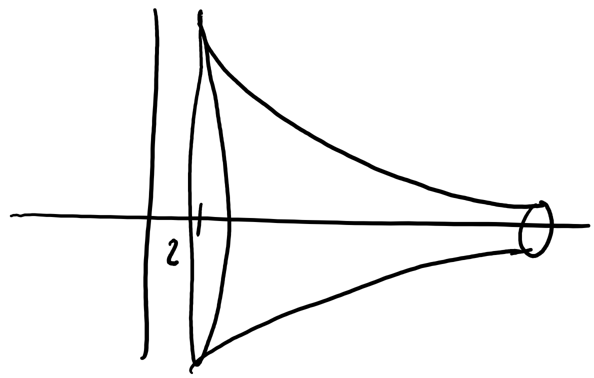
$$\int_0^{\pi} \frac{1}{\sqrt{x}}\, dx = \lim_{t \to 0^+} \int_t^{\pi} x^{-1/2}\, dx$$

$$= \lim_{t \to 0^+} \frac{2}{1} x^{1/2} \Big|_t^{\pi} = \lim_{t \to 0^+} 2\sqrt{\pi} - 2\sqrt{t}$$

$$\int_0^{\pi} \frac{1}{\sqrt{x}}\, dx = 2\sqrt{\pi}$$ converges

Thus by comp. Thrm $\displaystyle\int_0^{\pi} \frac{\sin(x)}{\sqrt{x}}\, dx$ will conv.

Example: Consider the function $f(x) = \dfrac{1}{x}$ is rotated arround the $x - axis$ for the interval $x \geq 2$.



Volume of the solid:

$$V = \int_2^\infty \pi \left(\frac{1}{x}\right)^2 \, dx = \int_2^\infty \frac{\pi}{x^2} \, dx$$

p-integral
p=2
conv.

Surface area of the solid:

$$SA = \int_2^\infty 2\pi \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx = 2\pi \int_2^\infty \frac{1}{x} \sqrt{\frac{x^4+1}{x^4}} \, dx$$

$$SA = 2\pi \int_2^\infty \frac{1}{x} \frac{\sqrt{x^4+1}}{x^2} \, dx = 2\pi \int_2^\infty \frac{\sqrt{x^4+1}}{x^3} \, dx$$

$$\int_2^\infty \frac{1}{x} \, dx$$

p-integral
p=1
diverge

Now:

$$\begin{aligned} x^4 &< x^4 + 1 \\ \sqrt{x^4} &< \sqrt{x^4+1} \\ x^2 &< \sqrt{x^4+1} \\ \frac{x^2}{x^3} &< \frac{\sqrt{x^4+1}}{x^3} \\ \frac{1}{x} &< \frac{\sqrt{x^4+1}}{x^3} \end{aligned}$$