Section 5.5: The Substitution Rule

Knowing $f(x) = (x^4+3)^4$ and $f'(x) = 4(x^4+3)^3 * 4x^3 = 16x^3(x^4+3)^3$

Compute $\int 16x^3(x^4+3)^3 dx =$ $(x^4+3)^4 + C$

Example: Compute.

$\frac{1}{9}\int 9 \cdot 2x(x^2+5)^8\, dx =$ $\boxed{\frac{1}{9}(x^2+5)^9 + C}$

$9(x^2+5)^8 \cdot 2x$

The substitution Rule If $\boxed{u = g(x)}$ is a differentiable function whose range is an interval I and f is continuous on I, then

$du = g'(x)\,dx$

$\int f(g(x))g'(x)\,dx = \int f(u)\,du$

Example: Compute the following.

A) $\int \cos(kx)\,dx = \int \frac{1}{k}\cos(u)\,du = \frac{1}{k}\sin(u) + C$

$u = kx$

$du = k\,dx$

$\frac{1}{k}\,du = dx$

$= \frac{1}{k}\sin(kx) + C$

$\int \cos(3x)\,dx = \frac{1}{3}\sin(3x) + C$

$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$

B) $\int \frac{12x^3+9}{(x^4+3x)^5}\,dx = \int \frac{3(4x^3+3)}{(x^4+3x)^5}\,dx$

$u = \text{junk}$

$(\text{junk})^{\text{power}}$

$\frac{\text{something}}{\text{junk}}$

e^{junk}

$\text{trig}(\text{junk})$

$u = x^4+3x$

$du = (4x^3+3)\,dx$

$= \int \frac{3}{u^5}\,du$

$\frac{1}{4x^3+3}\,du = dx$

$\int \frac{12x^3+9}{(x^4+3x)^5}\,dx = \int \frac{12x^3+9}{u^5} \cdot \frac{1}{4x^3+3}\,du$

$= \int \frac{3(4x^3+3)}{u^5} \cdot \frac{1}{4x^3+3}\,du$

$= \int \frac{3}{u^5}\,du = \int 3u^{-5}\,du$

$= \frac{3u^{-4}}{-4} + C = \boxed{\frac{-3}{4(x^4+3x)^4} + C}$

C) $\int \frac{e^{2+\sqrt{x}}}{\sqrt{x}}\,dx$ $=$ $\int 2\,e^{u}\,du = 2e^{u}+c$

$= \underbrace{2\,e^{2+\sqrt{x}}+c}$

$u = 2+\sqrt{x}$

$du = \frac{1}{2}\,x^{-1/2}\,dx$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \underbrace{\frac{1}{\sqrt{x}}\,dx}$

D) $\int x(x-8)^8\,dx = \int x \cdot u^8\,du = \int (u+8)u^8\,du$

$\boxed{u = x-8} \rightsquigarrow u+8=x$

$du = dx$

$= \int u^9 + 8u^8\,du$

$= \frac{1}{10}u^{10} + \frac{8}{9}u^9 + C$

$= \frac{1}{10}(x-8)^{10} + \frac{8}{9}(x-8)^9 + C$

$$\boxed{\int \frac{1}{1+x^2}\,dx = \arctan(x) + C}$$

E) $\int \frac{1+4x}{1+x^2}\,dx$

$u = 1+x^2$

$du = 2x\,dx$

$dx = \frac{1}{2x}\,du$

$$\int \frac{1+4x}{u} \cdot \frac{1}{2x}\,du$$

$$\int \frac{1}{1+x^2} + \frac{4x}{1+x^2}\,dx$$

$$= \int \frac{4x}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx$$

$$= \int \frac{4x}{u} \cdot \frac{1}{2x}\,du + \arctan(x) + C$$

$$= \int \frac{2}{u}\,du + \arctan(x) + C$$

$$= 2\ln|u| + \arctan(x) + C$$

$$= \underbrace{2\ln|1+x^2| + \arctan(x) + C}$$

F) $\int \frac{1}{x\ln(x)}\,dx$

$u = x\ln(x)$

$du = \left(\ln(x) + x \cdot \frac{1}{x}\right) dx$

$du = (\ln(x) + 1)\, dx$

$u = \ln(x)$

$du = \frac{1}{x}\, dx$

$x\,du = dx$

$$\int \frac{1}{x\ln(x)}\,dx = \int \frac{x\,du}{x\,u} = \int \frac{1}{u}\,du =$$

$$= \ln|u| + C = \ln|\ln(x)| + C$$

$u = g(x)$

The substitution Rule for Definite Integrals If $g'(x)$ is differentiable on $[a,b]$ and f is continuous on the range of g, then continuous on I, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$x = a \longrightarrow u = g(a)$

$x = b \longrightarrow u = g(b)$

Example: Compute

$$\int_0^2 x\cos(4x^2-1)dx =$$

$u = 4x^2 - 1$

$du = 8x\,dx$

$\frac{1}{8x}du = dx$

$x = 0 \longrightarrow u = -1$

$x = 2 \longrightarrow u = 16 - 1 = 15$

$$\int_0^2 x\cos(4x^2-1)dx = \int_{-1}^{15} x\cos(u)\frac{1}{8x}du$$

$$= \int_{-1}^{15} \frac{1}{8}\cos(u)\,du = \frac{1}{8}\sin(u)\Big|_{-1}^{15}$$

$$= \frac{1}{8}\sin(15) - \frac{1}{8}\sin(-1)$$

Example: Compute

$$\int_0^3 2x^3(1-x^2)^5\,dx = \int_{x=0}^{x=3} 2x^3u^5 \cdot \frac{-1}{2x}\,du = \int_{x=0}^{x=3} -x^2u^5\,du$$

$$\boxed{u = 1-x^2}$$

$du = -2x\,dx$ $\quad$ $x^2 = 1-u$

$-\frac{1}{2x}\,du = dx$

$$= \int_{x=0}^{x=3} -(1-u)u^5\,du$$

$x=0 \quad u=1$

$x=3 \quad u=-8$

$$= \int_{x=0}^{x=3} -u^5+u^6\,du$$

$$= -\frac{u^6}{6}+\frac{u^7}{7}\Big|_{x=0}^{x=3}$$

$$= -\frac{(1-x^2)^6}{6}+\frac{(1-x^2)^7}{7}\Big|_{x=0}^{x=3}$$

$$= -\frac{(-8)^6}{6}+\frac{(-8)^7}{7}-\left(-\frac{(1)^6}{6}+\frac{1^7}{7}\right)$$

$$= -\frac{8^6}{6}-\frac{8^7}{7}+\frac{1}{6}-\frac{1}{7}$$

$$\int_1^{-8} -u^5+u^6\,du = -\frac{u^6}{6}+\frac{u^7}{7}\Big|_1^{-8}$$

$$= -\frac{(-8)^6}{6} + \frac{(-8)^7}{7} - \left(-\frac{1}{6} + \frac{1}{7}\right)$$