Section 11.8: Power Series

Definition: A **power series** centered at $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

where x is a variable and c_n are constants called the coefficients of the series.

Example: Where is this power series centered? $X = 5$

$$\sum_{n=0}^{\infty}(2x-10)^n = \sum_{n=0} [2(x-5)]^n = \sum_{n=0} 2^n (x-5)^n$$

Example: Is the following a power series? yes

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

geometric power series

centered at $x=0$ $(a=0)$

$a=1$ conv. $|x|<1$ $\text{Sum} = \frac{a}{1-r}$

$r = x$ $\text{Sum} = \frac{1}{1-x}$

$$\sum_{n=0} x^n = \frac{1}{1-x} \quad \text{if } |x|<1$$

$$\sum_{n=0} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

Theorem: For a given power series, $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities for convergence.

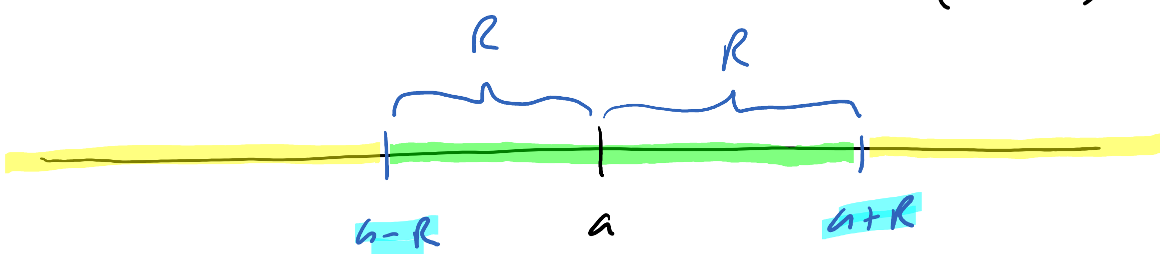
(i) The series converges only when $x = a$ $I = \{a\}$ $R = 0$

(ii) The series converges for all x. $I: (-\infty, \infty)$ $R = \infty$

(iii) The is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Radius $= R$

$$-R < x - a < R$$
$$a - R < x < a + R$$
$$I: (a-R, a+R)$$



The powers series is guarenteed to converge for all $a-R < x < a+R$

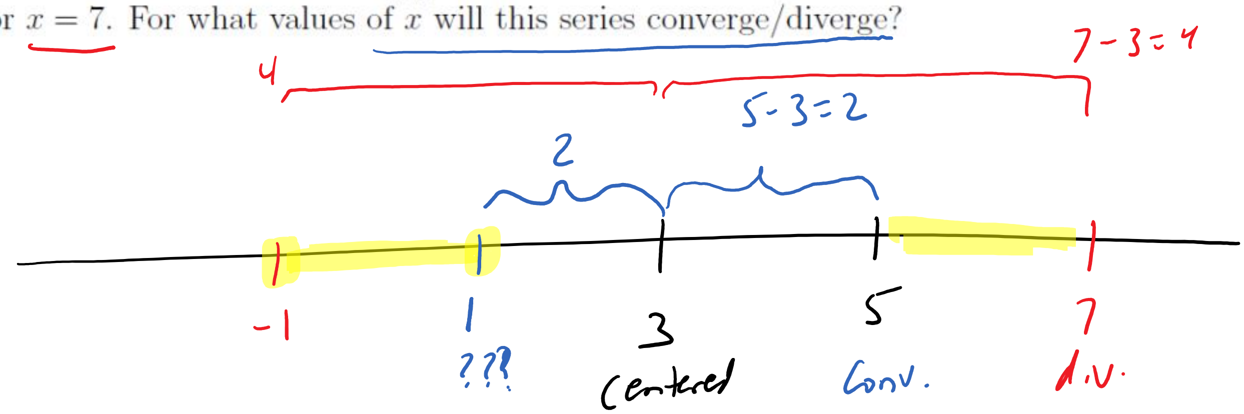
$\boxed{\text{The series will diverge on the Intervals } (a+R, \infty) \text{ and } (-\infty, a-R)}$

The convergence or divergence at $x = a + R$ . .

and $x = a - R$ needs to be tested.

Centered at $a = 3$ $(x = 3)$

Example: Suppose that the series $\sum_{n=0}^{\infty} c_n(x-3)^n$ converges for $x = 5$ and diverges for $x = 7$. For what values of x will this series converge/diverge?



Radius of conv $R \geq 2$ Know the values of x that converge are $1 < x \leq 5 \longrightarrow (1, 5]$

Radius of conv $2 \leq R \leq 4$

Know that x values that diverge are $x \geq 7$ and $x < -1$

for $5 < x < 7$ and $-1 \leq x \leq 1$ we do not have enough information to decide conv. or div.

$|x-a| < R$

Example: Find the radius and the interval of convergence for the power series.

$$0+\frac{1}{5}x+\frac{2}{5^2}x^2+\frac{3}{5^3}x^3+\frac{4}{5^4}x^4+\ldots = \sum_{n=0}^{\infty}\frac{n\,x^n}{5^n}$$

Centered at $x=0$

Ratio test

$$\lim_{n\to\infty}\left|a_{n+1}\cdot\frac{1}{a_n}\right|$$

$$a_n = \frac{n\,x^n}{5^n} \qquad a_{n+1} = \frac{(n+1)\,x^{n+1}}{5^{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{(n+1)\,x^{n+1}}{5^{n+1}}\cdot\frac{5^n}{n\,x^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{x^{n+1}\,5^n}{x^n\,5^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{\cancel{x^n}\cdot x^1\cdot\cancel{5^n}}{\cancel{x^n}\,\cancel{5^n}\cdot 5^1}\right|$$

$$= \lim_{n\to\infty}\left|\boxed{\frac{n+1}{n}}\cdot\frac{x}{5}\right| = \left|\frac{x}{5}\right| < 1$$

Then conv.

Centered at $a=0$

$|x-a| < R$

$$\left|\frac{x}{5}\right| < 1$$

$$\frac{|x|}{5} < 1$$

$$|x| < 5$$

$R = 5$

$-5 < x < 5$

$$R = \frac{5 - -5}{2} = \frac{10}{2} = 5$$

$$R = 5 - 0 = 5$$

start interval of conv.

start interval of conv. $\underbrace{-5 < x < 5}$

$$\boxed{\sum_{n=0} \frac{n\,x^n}{5^n}}$$

now we test the points.

$x = 5$

$$\sum_{n=0} \frac{n\,5^n}{5^n} = \sum_{n=0} n$$

div. by the test for div.

$x = -5$

$$\sum_{n=0} \frac{n\,(-5)^n}{5^n} = \sum \frac{n\,(-1)^n 5^n}{5^n} = \sum (-1)^n n$$

div. by Test for div.

Answer.

$R = 5$ $\quad I: (-5, 5) \to -5 < x < 5$

Example: Find the radius and the interval of convergence for the power series.

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

$a_n = \dfrac{x^n}{(n+1)!}$ $\qquad a_{n+1} = \dfrac{x^{n+1}}{(n+2)!}$

Ratio test

$$\lim_{n\to\infty} \left| \underbrace{\frac{x^{n+1}}{(n+2)!}}_{a_{n+1}} \cdot \underbrace{\frac{(n+1)!}{x^n}}_{\frac{1}{a_n}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{x^n}\cdot x}{\cancel{x^n}} \cdot \frac{\cancel{(n+1)!}}{(n+2)\cdot\cancel{(n+1)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n+2} \right| = 0$$

(This is true for all values of x)

Thus we conv. for all x-values.

$R = \infty \qquad I: (-\infty, \infty)$

centered at $x=4$

Example: Find the radius and the interval of convergence for the power series.

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{\sqrt{n}} \qquad a_n = \frac{(x-4)^n}{\sqrt{n}} \qquad a_{n+1} = \frac{(x-4)^{n+1}}{\sqrt{n+1}}$$

Ratio test

$$\lim_{n\to\infty} \left| \underbrace{\frac{(x-4)^{n+1}}{\sqrt{n+1}}}_{a_{n+1}} \cdot \underbrace{\frac{\sqrt{n}}{(x-4)^n}}_{\frac{1}{a_n}} \right| = \lim_{n\to\infty} \left| (x-4) \frac{\sqrt{n}}{\sqrt{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (x-4) \sqrt{\boxed{\frac{n}{n+1}}} \right| = |x-4|$$

Conv. if $\underbrace{|x-4| < 1}$

$|x-4| < R$

$R = 1$

starting Interval

$-1 < x-4 < 1$

$4-1 < x < 4+1$

$3 < x < 5$

$$\sum \frac{(x-4)^n}{\sqrt{n}}$$

Test the end points

Do not ever do this

$|x-4| < 1$

$|x| < 5$

$-5 < x < 5$

$\underline{\underline{X=5}}$ $\quad \sum_{n=1} \frac{(5-4)^n}{\sqrt{n}} = \sum \frac{1^n}{\sqrt{n}}$

$= \sum \frac{1}{\sqrt{n}}$ $\quad$ p-series $p = \frac{1}{2}$

div.

$\underline{X=3}$ $\quad \sum_{n=1} \frac{(3-4)^n}{\sqrt{n}} = \sum_{n=1} \frac{(-1)^n}{\sqrt{n}}$

use AST

$b_n = \frac{1}{\sqrt{n}}$

$n=0$

$\lim_{n\to\infty} b_n = 0$

Conv. by AST.

b_n are dec.

Answer: $R = 1$ $\quad I: [3, 5)$

Example: Find the radius and the interval of convergence for the power series.

$$\underbrace{\sum_{n=0}^{\infty} n!(x-1)^n}$$

centered at $x=1$

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,(x-1)^{n+1}}{n!\,(x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cdot n!}{n!}(x-1)\right|$$

$$= \lim_{n\to\infty}\left|(n+1)\cdot(x-1)\right| = \begin{cases} 0 & \text{if } x=1 \\ \infty & x\neq 1 \end{cases}$$

The series will only converge if $x=1$

Answer $R=0$ $I:\{1\}$

centered $x = \frac{4}{3}$

Example: Find the radius and the interval of convergence for the power series.

$$\sum_{n=0}^{\infty} \frac{n+1}{10^n}(3x-4)^n$$

a_{n+1} $\frac{1}{a_n}$

$$\lim_{n\to\infty}\left|\frac{(n+2)(3x-4)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(n+1)(3x-4)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+2}{n+1} \cdot \frac{(3x-4)^{n+1}}{(3x-4)^n} \cdot \frac{10^n}{10^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\boxed{\frac{n+2}{n+1}} \cdot \frac{3x-4}{10}\right| = \left|\frac{3x-4}{10}\right| < 1$$

$|x-a| < R$

$$\frac{|3x-4|}{10} < 1 \qquad |3x-4| < 10$$

$$|3(x-\tfrac{4}{3})| < 10$$

$$|x-\tfrac{4}{3}| < \underbrace{\frac{10}{3} = R}$$

$$-10 < 3x-4 < 10$$

$$-6 < 3x < 14$$

$$\underbrace{-2 < x < \frac{14}{3}}$$

Starting Interval

centered at $a = \frac{4}{3}$

$R = \frac{14}{3} - \frac{4}{3} = \frac{10}{3}$

Test end points

$x = \frac{14}{3}$

$$\sum \frac{n+1}{10^n}\left(3\cdot\frac{14}{3} - 4\right)^n = \sum \frac{n+1}{10^n}(10)^n$$

$$= \sum n+1$$

div. by test for div.

$x = -2$

$$\sum \frac{n+1}{10^n}\left(3(-2)-4\right)^n = \sum \frac{n+1}{10^n}(-10)^n$$

$$= \sum (n+1)(-1)^n$$

div. by the Test for div.

Answer $\quad R = \frac{10}{3} \quad I : \left(-2, \frac{14}{3}\right)$