Application of Taylor Polynomials

**Taylor Polynomials.**

The Taylor series of a function, $f(x)$, can be expressed: $\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.}$

The **n-th degree Taylor polynomial** of $f(x)$ at $a$, denoted $T_n$ is given by

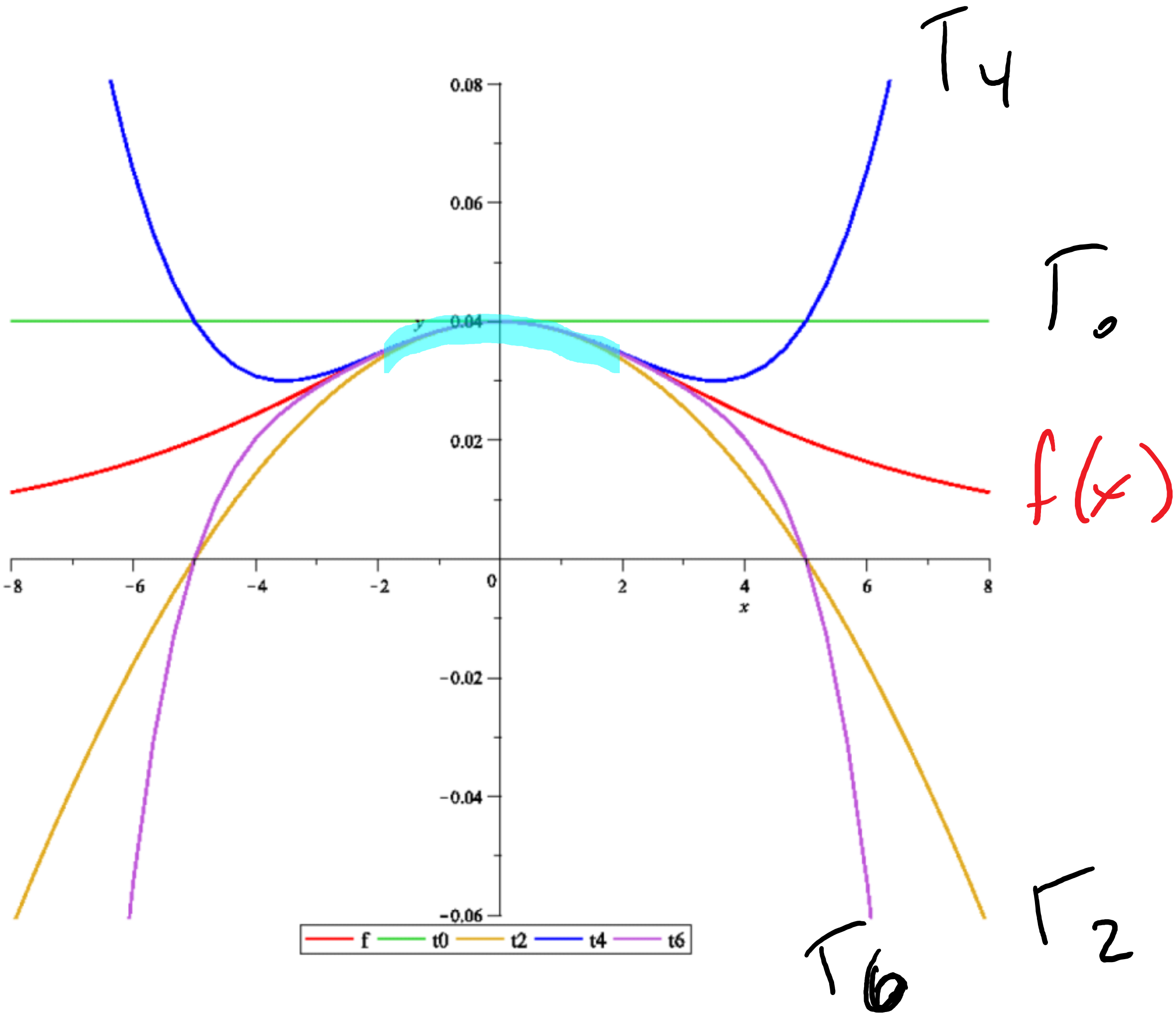
$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

degree n

$T_1$: 1st degree taylor polynomial

equation of the tangent line.

The following graph shows the function $f(x) = \dfrac{10}{x^2+25}$ and $T_0$, $T_2$, $T_4$, and $T_6$.



$$C_n = \frac{f^{(n)}(2)}{2!}$$

Example: Find the Taylor polynomials, $T_1$, $T_2$, and $T_3$, for $f(x) = xe^x$ centered at $a = 2$.

$$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + C_4(x-a)^4 + \cdots$$

$$f(x) = xe^x$$
$$f'(x) = 1e^x + x \cdot e^x = (1+x)e^x$$
$$f''(x) = 1e^x + (1+x)e^x = (2+x)e^x$$
$$f'''(x) = 1e^x + (2+x)e^x = (3+x)e^x$$

$$\boxed{f^{(n)}(x) = (n+x)e^x}$$

$$f(2) = 2e^2$$
$$f'(2) = 3e^2$$
$$f''(2) = 4e^2$$
$$f'''(2) = 5e^2$$

$$T_1 = C_0 + C_1(x-2) = 2e^2 + \frac{3e^2}{1!}(x-2)$$

$$\boxed{T_1 = 2e^2 + 3e^2(x-2)}$$

$$T_2 = C_0 + C_1(x-2) + C_2(x-2)^2$$

$$T_2 = 2e^2 + 3e^2(x-2) + \frac{4e^2}{2!}(x-2)^2$$

$$\boxed{T_2 = 2e^2 + 3e^2(x-2) + 2e^2(x-2)^2}$$

$$T_3 = C_0 + C_1(x-2) + C_2(x-2)^2 + C_3(x-2)^3$$

$$T_3 = c_0 + c_1 (x-2) + c_2 (x-2)^2 + c_3 (x-2)^3$$

$$= 2e^2 + 3e^2 (x-2) + 2e^2 (x-2)^2 + \frac{5e^2}{3!} (x-2)^3$$

$$T_3 = 2e^2 + 3e^2 (x-2) + 2e^2 (x-2)^2 + \frac{5e^2}{6} (x-2)^3$$

Example: Find the Taylor polynomials, $T_1$, $T_4$, $T_5$, and $T_7$ for $f(x) = \dfrac{x}{1+5x^3}$ centered at $a = 0$

$$f(x) = x \cdot \frac{1}{1-(-5x^3)} = x \sum_{n=0}^{\infty} (-5x^3)^n$$

$$= x \sum_{n=0} (-1)^n 5^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n 5^n x^{3n+1}$$

$$f(x) = x - 5x^4 + 25x^7 - 125x^{10} + \cdots\cdots$$

$$T_1 = x$$

$$T_4 = x - 5x^4$$

$$T_5 = x - 5x^4 + 0x^5 = x - 5x^4 = T_4$$

$$T_7 = x - 5x^4 + 25x^7$$

Example: Express $f(x) = 2x^3 + 4x^2 + 7x + 6$ as a Taylor polynomial(series) about $a = 2$.

$$c_n = \frac{f^{(n)}(2)}{n!}$$

$c_n = 0$ if $n \geq 4$

$f(2) = 52$

$f'(x) = 6x^2 + 8x + 7 \qquad f'(2) = 47$

$f''(x) = 12x + 8 \qquad f''(2) = 32$

$f'''(x) = 12 \qquad f'''(2) = 12$

$f^{(4)}(x) = 0$

Taylor polynomial $= 52 + \frac{42}{1!}(x-2) + \frac{32}{2!}(x-2)^2 + \frac{12}{3!}(x-2)^3$

$T_3$

$$\boxed{T_3 = 52 + 42(x-2) + 16(x-2)^2 + 2(x-2)^3}$$