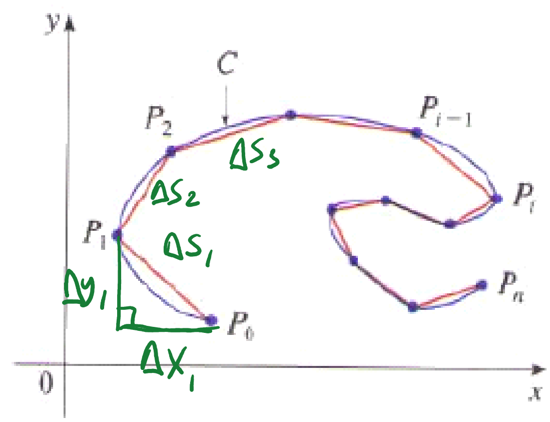
Section 10.2: Calculus with Parametric Functions.

Arc Length

Suppose that C is a smooth curve defined by $x = f(t)$ and $y = g(t)$ for $[a, b]$. Let $\{P_i\}$ be a set of points on the curve that partition of the interval $[a, b]$ such that Δt is equal for each subinterval.



Then the length of the curve(arc length) is given by

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s_i = \int_a^b ds$$

$$\Delta s_i = |P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$\Delta s_i = \sqrt{(f'(t_i)\Delta t)^2 + (g'(t_i)\Delta t)^2}$$

$$\Delta s_i = \sqrt{(f'(t_i))^2 + (g'(t_i))^2}\,\Delta t$$

$$\frac{\Delta x}{\Delta t} \approx \frac{dx}{dt}$$

$$\Delta x \approx \frac{dx}{dt}\Delta t$$

$$\Delta x \approx f'(t)\,\Delta t$$

$$\Delta y \approx g'(t)\,\Delta t$$

$L = \int_a^b ds$ where $ds = \sqrt{(f'(t))^2 + (g'(t))^2}\,dt$

$$ds = \sqrt{(x')^2 + (y')^2}\,dt$$

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Example: Find the length of the arc of the curve given by $x(t) = 3t - t^3$, $y(t) = 3t^2$ from the point $(0, 0)$ to the point $(-2, 12)$ $t=2$

$3t^2 = 0$

$t = 0$

$3t^2 = 12$

$t^2 = 4$

$t = \pm 2$

$x(2) = 6 - 8 = -2$ ✓

$x(-2) = -6 - (-8) = 2$ ✗

$x' = 3 - 3t^2$

$y' = 6t$

$$L = \int_0^2 \sqrt{(3-3t^2)^2 + (6t)^2}\, dt$$

$$L = \int_0^2 \sqrt{9 - 18t^2 + 9t^4 + 36t^2}\, dt$$

$$L = \int_0^2 \sqrt{9 + 18t^2 + 9t^4}\, dt$$

$$= \int_0^2 \sqrt{(3+3t^2)^2}\, dt = \int_0^2 3 + 3t^2\, dt$$

$$= 3t + t^3 \Big|_0^2 = 6 + 8 = 14$$

Example: Find the length of the arc of the curve $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$, on the interval $1 \le x \le 2$.

$$X = t \qquad y = \frac{t^3}{6} + \frac{1}{2t} \qquad 1 \le t \le 2$$

$$X' = 1 \qquad y' = \frac{3t^2}{6} - \frac{1}{2t^2} = \frac{t^2}{2} - \frac{1}{2t^2}$$

$$L = \int_1^2 \sqrt{(1)^2 + \left(\frac{t^2}{2} - \frac{1}{2t^2}\right)^2}\, dt$$

$$= \int_1^2 \sqrt{1 + \frac{t^4}{4} - 2\left(\frac{t^2}{2}\right)\left(\frac{1}{2t^2}\right) + \frac{1}{4t^4}}\, dt$$

$$= \int_1^2 \sqrt{1 + \frac{t^4}{4} - \frac{1}{2} + \frac{1}{4t^4}}\, dt$$

$$= \int_1^2 \sqrt{\frac{t^4}{4} + \frac{1}{2} + \frac{1}{4t^4}}\, dt$$

$$= \int_1^2 \sqrt{\left(\frac{t^2}{2} + \frac{1}{2t^2}\right)^2}\, dt = \int_1^2 \frac{t^2}{2} + \frac{1}{2t^2}\, dt$$

$$= \int_1^2 \sqrt{\left(\frac{t^2}{2} + \frac{1}{2t^2}\right)^2}\, dt = \int_1^2 \frac{t^2}{2} + \frac{1}{2t^2}\, dt$$

$$\frac{1}{2}t^{-2}$$

$$= \left(\frac{t^3}{6} + \frac{1}{2}\frac{t^{-1}}{(-1)}\right)\Bigg|_1^2$$

$$= \left(\frac{t^3}{6} - \frac{1}{2t}\right)\Bigg|_1^2 = \frac{8}{6} - \frac{1}{4} - \left(\frac{1}{6} - \frac{1}{2}\right)$$

$$= \frac{17}{12}$$

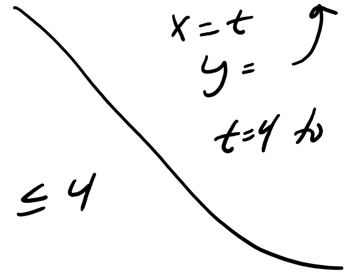
$y = \sqrt[3]{(5-x)^2}$

$x = t$

$y =$

$t=4$ to $t=-3$

Example: Find the length of the arc of the curve $x = 5 - \sqrt{y^3}$, from the point $(4, 1)$ to the point $(-3, 4)$



$x = 5 - t^{3/2}$ $\quad y = t$ $\quad 1 \le t \le 4$

$x' = -\frac{3}{2} t^{1/2}$ $\quad y' = 1$

$$L = \int_1^4 \sqrt{\left(-\frac{3}{2} t^{1/2}\right)^2 + (1)^2}\, dt$$

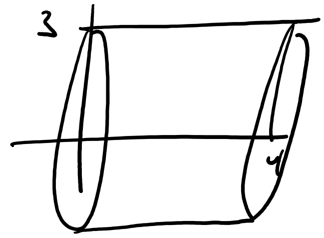
$$= \int_1^4 \sqrt{\frac{9}{4} t + 1}\, dt$$

$u = \frac{9}{4} t + 1$

$$= \dots = \frac{8}{27}(10)^{3/2} - \frac{8}{27}(3.25)^{3/2}$$

Surface Area

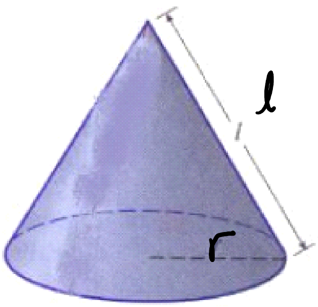
Rotate $y = 3$ from $x = 0$ to $x = 4$ about the x-axis. Find the surface area of the object.



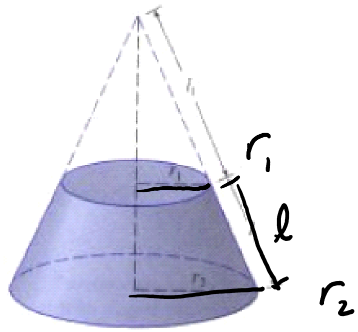
$2\pi r h = 2\pi(3)(4)$

$= 24\pi$

Surface Area of cones.



$SA = \pi r l$



$SA = \pi(r_1 + r_2)l$

let

$r = \frac{1}{2}(r_1 + r_2)$

$2r = r_1 + r_2$

$SA = 2\pi r l$

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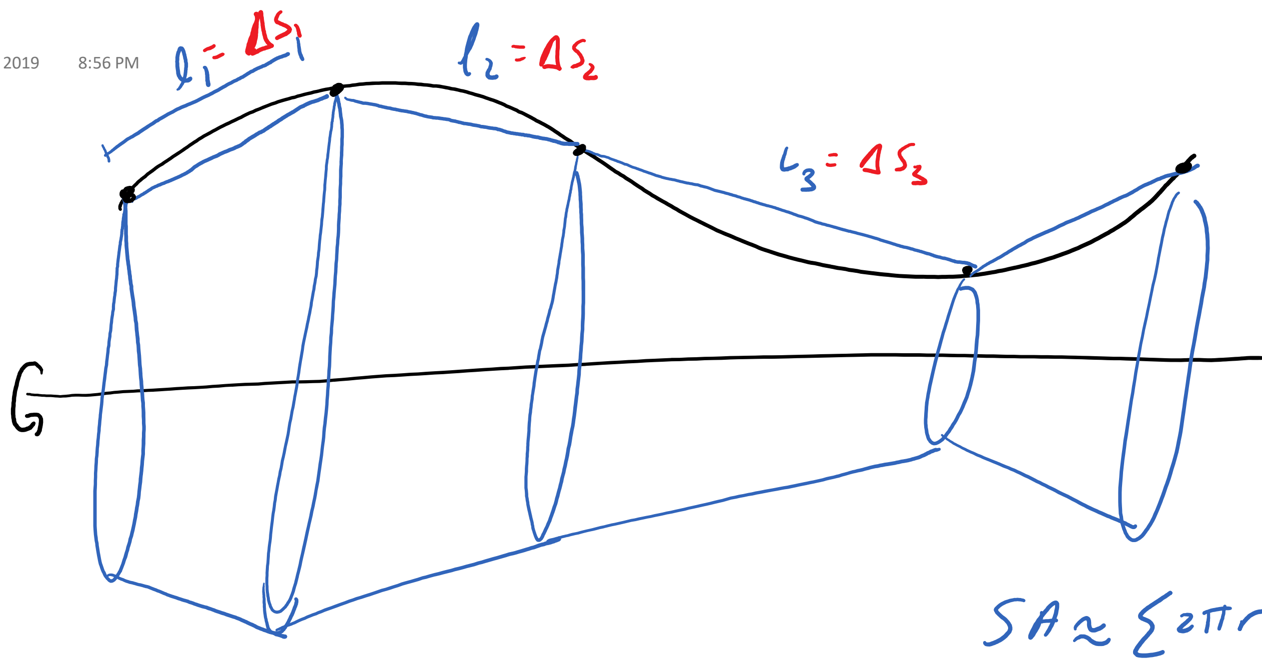

$SA \approx \sum 2\pi r \underline{\underline{l}}$

with a limit

$SA = \int_a^b 2\pi r \, ds$

The surface area of a curve rotated about the y-axis:

$$SA = \int_a^b 2\pi x \, ds$$

The surface area of a curve rotated about the x-axis:

$$SA = \int_a^b 2\pi y \, ds$$

Example: Find the area of the surface obtained by rotating the curve $y = \sqrt{x}$ from the point $(1, 1)$ to $(4, 2)$ about the x-axis.

$1 \le t \le 4$

$x = t$

$x' = 1$

$y = t^{1/2}$

$y' = \frac{1}{2} t^{-1/2} = \frac{1}{2\sqrt{t}}$

$$SA = \int_a^b 2\pi y \, ds$$

$$SA = \int_1^4 2\pi \sqrt{t} \sqrt{(1)^2 + \left(\frac{1}{2\sqrt{t}}\right)^2} \, dt$$

$$= \int_1^4 2\pi \sqrt{t} \sqrt{1 + \frac{1}{4t}} \, dt$$

$$= \int_1^4 2\pi \sqrt{t\left(1 + \frac{1}{4t}\right)} \, dt$$

$$= \int_1^4 2\pi \sqrt{t + \frac{1}{4}} \, dt$$

$u = t + \frac{1}{4}$

$$= \cdots = \frac{4\pi}{3}\left[(4.25)^{3/2} - (1.25)^{3/2}\right]$$

Example: Find the area of the surface obtained by rotating the curve $x = t$, $y = \frac{t^2}{4} - \frac{\ln(t)}{2}$ on the interval $1 \le t \le 4$ about the y-axis.

$$x' = 1 \qquad y' = \frac{2t}{4} - \frac{1}{2t} = \frac{t}{2} - \frac{1}{2t}$$

$$SA = \int_1^4 2\pi x \, ds = \int_1^4 2\pi t \sqrt{(1)^2 + \left(\frac{t}{2} - \frac{1}{2t}\right)^2} \, dt$$

$$= \int_1^4 2\pi t \sqrt{1 + \frac{t^2}{4} - 2\left(\frac{t}{2}\right)\left(\frac{1}{2t}\right) + \frac{1}{4t^2}} \, dt$$

$$= \int_1^4 2\pi t \sqrt{1 + \frac{t^2}{4} - \frac{1}{2} + \frac{1}{4t^2}} \, dt$$

$$= \int_1^4 2\pi t \sqrt{\frac{t^2}{4} + \frac{1}{2} + \frac{1}{4t^2}} \, dt$$

$$= \int_1^4 2\pi t \sqrt{\left(\frac{t}{2} + \frac{1}{2t}\right)^2} \, dt$$

$$= \int_1^4 2\pi t \left(\frac{t}{2} + \frac{1}{2t}\right) dt = \int_1^4 2\pi \left(\frac{t^2}{2} + \frac{1}{2}\right) dt$$

$$= \pi \int_1^4 t^2 + 1 \, dt = \cdots\cdots = 24\pi$$