Section 7.8: Improper Integrals

$$\int_a^{\infty} f(x)\,dx \quad \text{or} \quad \int_{-\infty}^{a} f(x)\,dx$$

Improper integrals of Type I

(a) If $\int_a^t f(x)\,dx$ exist for every number $t \geq a$ then $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$ provided this limit exists (as a finite number).

(b) If $\int_t^a f(x)\,dx$ exist for every number $t \leq a$ then $\int_{-\infty}^{a} f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx$ provided this limit exists (as a finite number).

The improper integrals in (a) and (b) are called convergent if the limits exists and divergent if the limit does not exist.

(c) If both $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{a} f(x)dx$ are convergent, then we define $\int_{-\infty}^{\infty} f(x)\,dx = \int_a^{\infty} f(x)\,dx + \int_{-\infty}^{a} f(x)\,dx$

Compute these integrals.

A) $\int_2^{\infty} \frac{1}{(x+3)^{1.5}}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{(x+3)^{1.5}}\,dx = \lim_{t\to\infty} \left. \frac{-2}{\sqrt{x+3}} \right|_2^t$

$$\int \frac{1}{(x+3)^{1.5}}\,dx = \int \frac{1}{u^{1.5}}\,du$$

$u = x+3$

$du = dx$

$$= \int u^{-1.5}\,du$$

$$= \frac{u^{-.5}}{-.5} = \frac{u^{-.5}}{-\frac{1}{2}}$$

$$= -2u^{-.5} = \frac{-2}{\sqrt{u}}$$

$$= \frac{-2}{\sqrt{x+3}}$$

$$= \lim_{t\to\infty}\left(\underbrace{\frac{-2}{\sqrt{t+3}}}_{\to 0} - \frac{-2}{\sqrt{5}} \right)$$

$$= 0 + \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$\int_2^{\infty} \frac{1}{(x+3)^{1.5}}\,dx$ converges.

and it converges to the # $\frac{2}{\sqrt{5}}$

B) $\displaystyle\int_0^\infty \frac{\cos(x)}{1+\sin^2(x)}\,dx = \lim_{t\to\infty}\int_0^t \frac{\cos x}{1+\sin^2 x}\,dx = \lim_{t\to\infty} \arctan(\sin(x))\Big|_0^t$

$$\int \frac{\cos x}{1+\sin^2(x)}\,dx = \int \frac{1}{1+u^2}\,du$$

$u = \sin(x)$

$du = \cos x\,dx$

$= \arctan(u)$

$= \arctan(\sin(x))$

$$= \lim_{t\to\infty}\left[\arctan(\sin(t)) - \arctan(\sin(0))\right]$$

$= \text{DNE}$

This improper Integral diverges

C) $\displaystyle\int_1^{\infty} \frac{x+1}{(x+3)(x+4)}\,dx = \lim_{t\to\infty} \int_1^t \frac{x+1}{(x+3)(x+4)}\,dx = \lim_{t\to\infty} \Big(3\ln|x+4| - 2\ln(x+3)\Big)\Big|_1^t$

$$\int \frac{x+1}{(x+3)(x+4)}\,dx$$

$$\frac{x+1}{\boxed{(x+3)(x+4)}} = \frac{A}{x+3} + \frac{B}{x+4}$$

$$(x+1) = A(x+4) + B(x+3)$$

if $x=-4$: $\quad -3 = B(-1)$, $\quad B = 3$

if $x=-3$: $\quad -2 = A(1)$, $\quad A = -2$

$$\int \frac{-2}{x+3} + \frac{3}{x+4}\,dx = -2\ln|x+3| + 3\ln|x+4|$$

$$\lim_{t\to\infty}\Big[\underbrace{3\ln(t+4)}_{\infty} - \underbrace{2\ln(t+3)}_{\infty} - \big(3\ln(5) - 2\ln(4)\big)\Big]$$

$\infty - \infty$

$\ln(t+4)^3 - \ln(t+3)^2$

$$\lim_{t\to\infty} \ln\left(\frac{(t+4)^3}{(t+3)^2}\right) - \big(3\ln(5) - 2\ln(4)\big) = \infty$$

The Integral diverges to ∞ !! ☺

The Integral diverges to ∞ !!

$$\lim_{t\to\infty} \frac{(t+4)^3}{(t+3)^2} = \lim_{t\to\infty} \frac{t^3 + \cdots}{t^2+6t+9} \overset{L'H}{=} \infty$$

p- Integral

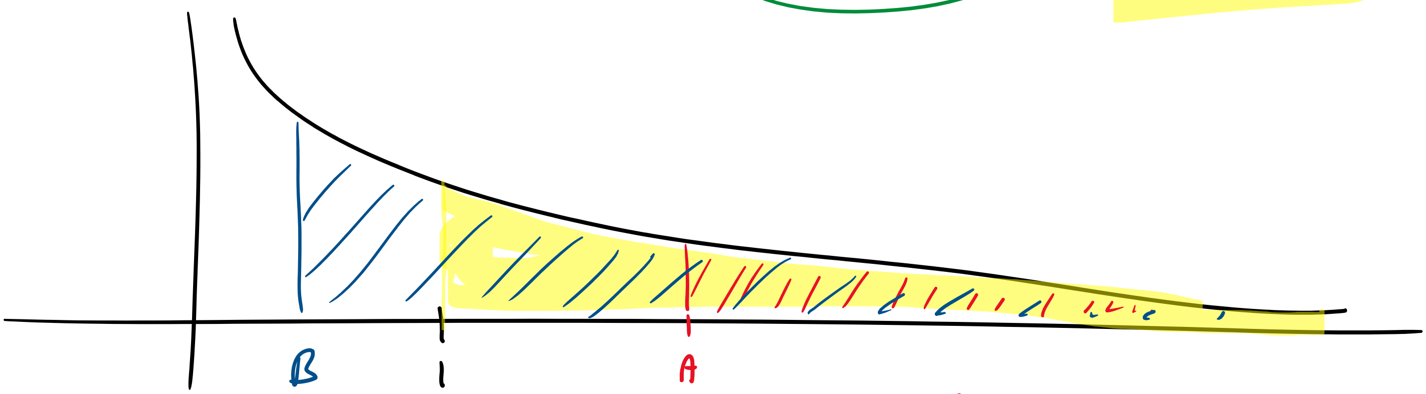
Fact: The $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and diverges if $p \le 1$.

Example: For what values of p will these integrals converge?

$$\int_1^\infty \frac{5}{x^p}\,dx \qquad \int_A^\infty \frac{5}{x^p}\,dx, \text{ where } A > 0.$$

if $p > 1$
These converge

$$\int_1^\infty \frac{1}{x^5}\,dx$$

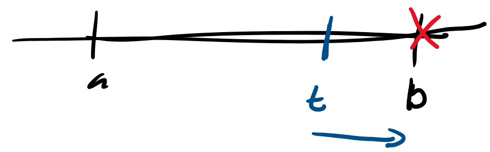


if $0 < B < 1$

$$\int_B^\infty \frac{1}{x^p}\,dx = \int_B^1 \frac{1}{x^p}\,dx + \int_1^\infty \frac{1}{x^p}\,dx$$

if $A > 1$ $\int_A^\infty \frac{1}{x^p}\,dx$

Page 6: Type II



Improper integrals of Type II

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then $\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx$

provided this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then $\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^b f(x)\, dx$

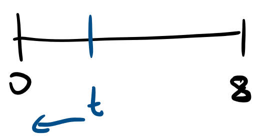
provided this limit exists (as a finite number).

The improper integrals in (a) and (b) are called convergent if the limits exists and divergent if the limit does not exist.

(c) If f has a discontinuity at c where $a < c < b$, and both $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ are convergent, then we define $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

Compute these integrals.

A) $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$



$\dfrac{1}{\sqrt[3]{x}}$ DNE for $x=0$ and this value is in our interval.

Integral is improper.

$$= \lim_{t\to 0^+} \int_t^8 x^{-1/3}\,dx = \lim_{t\to 0^+} \frac{3}{2}x^{2/3}\Big|_t^8$$

$$= \lim_{t\to 0^+}\left(\frac{3}{2}(8)^{2/3} - \underbrace{\frac{3}{2}(t)^{2/3}}_{\to 0}\right)$$

$$= \frac{3}{2}(8)^{2/3} = \frac{3}{2}\cdot 4 = 6$$

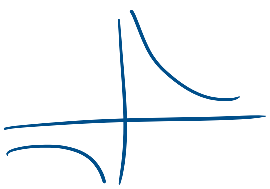
The Integral converges.

B) $\int_{-3}^{1} \frac{1}{x^2}\,dx = \int_{-3}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$

not continuous at $x=0$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-2}\,dx = \lim_{t\to 0^+} \frac{x^{-1}}{-1}\Bigg|_t^1 = \lim_{t\to 0^+} -\frac{1}{x}\Bigg|_t^1$$

$$= \lim_{t\to 0^+}\left(-1 - \underbrace{\frac{-1}{t}}_{+\infty}\right) = \lim_{t\to 0^+}\left(-1 + \frac{1}{t}\right) = \infty$$



The Integral diverges.

So $\int_{-3}^{1} \frac{1}{x^2}\,dx$ will also diverge.

Fact: The $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ is convergent if $p < 1$ and diverges if $p \geq 1$.

not the p-integral.

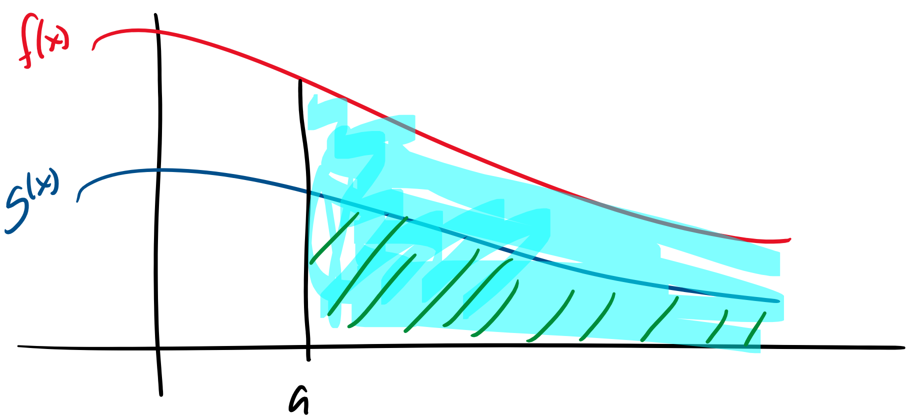
Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$

(a) If $\displaystyle\int_a^\infty f(x)dx$ is convergent, then $\displaystyle\int_a^\infty g(x)dx$ is convergent.

(b) If $\displaystyle\int_a^\infty g(x)dx$ is divergent, then $\displaystyle\int_a^\infty f(x)dx$ is divergent.

$$\int_a^\infty f(x)\,dx \qquad \int_a^\infty g(x)\,dx$$



Use the comparison theorem to decide if these integrals converge or diverge.

A) $\int_2^\infty \frac{1}{\sqrt[3]{x^2 - 1}}\,dx = J$

$0 < x^2 - 1 < x^2$

$\sqrt[3]{x^2-1} < \sqrt[3]{x^2}$

$\frac{1}{\sqrt[3]{x^2-1}} > \boxed{\frac{1}{\sqrt[3]{x^2}}}$

$\int_2^\infty \frac{1}{\sqrt[3]{x^2}}\,dx = \underbrace{\int_2^\infty \frac{1}{x^{2/3}}\,dx}$

p-integral.
$p = 2/3$

This improper integral diverges.

By the comparison thm J will also diverge.

Page 11

B) $\displaystyle\int_2^\infty \frac{1}{\sqrt[3]{x^4-1}}\,dx = K$

$x^4 - 1 < x^4$

$\sqrt[3]{x^4-1} < \sqrt[3]{x^4}$

$$\frac{1}{\sqrt[3]{x^4-1}} > \underbrace{\frac{1}{\sqrt[3]{x^4}}}$$

$$\int_2^\infty \frac{1}{x^{4/3}}\,dx = J$$

p-integral $p = 4/3$

Since $p > 1$ J will converge.

The smaller Integral (J) converges so we know nothing about K.

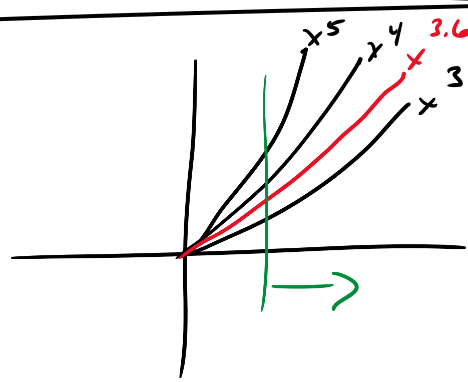
comparison test failed.

$x^{3.6} < x^4$

$x^{3.6} < x^4 - 1$

$x^{1.2} = x^{\frac{3.6}{3}} = \sqrt[3]{x^{3.6}} < \sqrt[3]{x^4-1}$

$$\underbrace{\frac{1}{x^{1.2}}} > \frac{1}{\sqrt[3]{x^4-1}}$$



$$\int_2^\infty \frac{1}{x^{1.2}}\,dx = L$$

p-integral $p = 1.2 > 1$ so L converges.

By the comparison Thrm K will also converges.

C) $\int_2^\infty e^{-x^4}\,dx = K$

$$x < x^4$$

$$-x > -x^4$$

$$e^{-x} > e^{-x^4}$$

$$J = \int_2^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_2^t e^{-x}\,dx$$

$$= \lim_{t\to\infty} -e^{-x}\Big|_2^t$$

$$= \lim_{t\to\infty} \left(-e^{-t} - -e^{-2}\right)$$

$$= \lim_{t\to\infty} \left(-\frac{1}{e^t} + e^{-2}\right) = 0 + e^{-2} = e^{-2}$$

J converges.

By comparison we know K also converges.

D) $\int_1^{\infty} \frac{3+2\cos(2x)}{x^2}\,dx = M$

$-1 \le \cos(2x) \le 1$

$-2 \le 2\cos(2x) \le 2$

$1 = 3 - 2 \le 3 + 2\cos(2x) \le 2 + 3 = 5$

$1 \le 3 + 2\cos(2x) \le 5$

$\frac{1}{x^2} \le \frac{3+2\cos(2x)}{x^2} \le \frac{5}{x^2}$

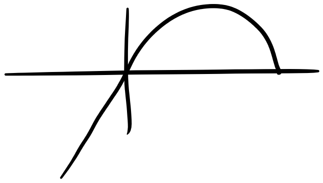
$J = \int_1^{\infty} \frac{1}{x^2}\,dx$ and $K = \int_1^{\infty} \frac{5}{x^2}\,dx$

Both are p-integrals and $p = 2 > 1$

So both J + K will converge.

Since K conv. By the comparison thrm M will also conv.

E) $\int_0^{\pi} \frac{\sin(x)}{\sqrt{x}}\, dx = K$



$0 \le \sin(x) \le 1$

$0 = \frac{0}{\sqrt{x}} \le \frac{\sin(x)}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ for $x \neq 0$.

$$J = \int_0^{\pi} \frac{1}{\sqrt{x}}\, dx = \lim_{t \to 0^+} \int_t^{\pi} x^{-1/2}\, dx = \lim_{t \to 0^+} \frac{2}{1} x^{1/2} \Big|_t^{\pi}$$

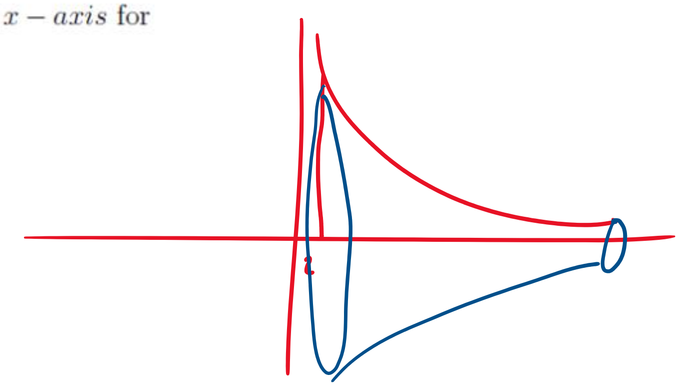
$$= \lim_{t \to 0^+} \left(2\sqrt{\pi} - 2\sqrt{t} \right) = 2\sqrt{\pi}$$

J converges.

By the comparison thrm K will converge.

Page 15: Something Interesting

Example: Consider the function $f(x) = \dfrac{1}{x}$ is rotated arround the $x-axis$ for the interval $x \geq 2$.



Volume of the solid:

$$V = \int_2^{\infty} \pi \left(\frac{1}{x}\right)^2 \, dx = \int_2^{\infty} \frac{\pi}{x^2} \, dx =$$

conv. since $p = 2 > 1$ p integral.

Surface area of the solid:

$$SA = \int_2^{\infty} 2\pi \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx = 2\pi \int_2^{\infty} \frac{1}{x} \sqrt{\frac{x^4+1}{x^4}} \, dx$$

$$SA = 2\pi \int_2^{\infty} \frac{1}{x} \frac{\sqrt{x^4+1}}{x^2} \, dx = 2\pi \int_2^{\infty} \frac{\sqrt{x^4+1}}{x^3} \, dx$$

diverges.

Now:

$$x^4 < x^4 + 1$$

$$\sqrt{x^4} < \sqrt{x^4+1}$$

$$x^2 < \sqrt{x^4+1}$$

$$\frac{x^2}{x^3} < \frac{\sqrt{x^4+1}}{x^3}$$

$$\frac{1}{x} < \frac{\sqrt{x^4+1}}{x^3}$$

$$\int_2^{\infty} \frac{1}{x} \, dx$$

p integral $p = 1$ diverges.