Section 11.1: Sequences

Definition: A **sequence** is a list of numbers written in a definite order.
$a_1, a_2, a_3, ...$ or $\{a_n\}_{n=1}^{\infty}$

Example: Find a general formula for these sequences.

A) $\left\{\frac{5}{9}, \frac{6}{16}, \frac{7}{25}, \frac{8}{36}, \ldots\right\}$

$3^2 \quad 4^2 \quad 5^2 \quad 6^2$

let $j = n-2 \rightarrow j = 1, 2, 3, \ldots$

$j + 2 = n$

$$a_j = \frac{(j+2)+2}{(j+2)^2} = \frac{j+4}{(j+2)^2}$$

$$a_n = \frac{n+2}{n^2} \quad \text{if } n \geq 3$$

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$$a_n = \boxed{} \quad n = 1, 2, \ldots\ldots$$

Rewrite with n.

$$a_n = \frac{n+4}{(n+2)^2} \quad n = 1, 2, 3, \ldots.$$

B) $\left\{\frac{3}{4}, \frac{6}{11}, \frac{9}{18}, \frac{12}{25}, \frac{15}{32}, \dots\right\}$

7 7 7 7

$a_n = \dfrac{3n}{7n-3}$ $n = 1, 2, 3 \dots$

Bottom formula is a line.

n

(1, 4)

(2, 11)

$m = \dfrac{11-4}{2-1} = \dfrac{7}{1} = 7$

$y - 4 = 7(x - 1)$

$y = 4 + 7x - 7$

$y = 7x - 3$

C) $\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$

Recursive seq.

$a_1 = 1$

$a_2 = 1$

$a_3 = 2$

$\vdots$

$a_1 = 1$

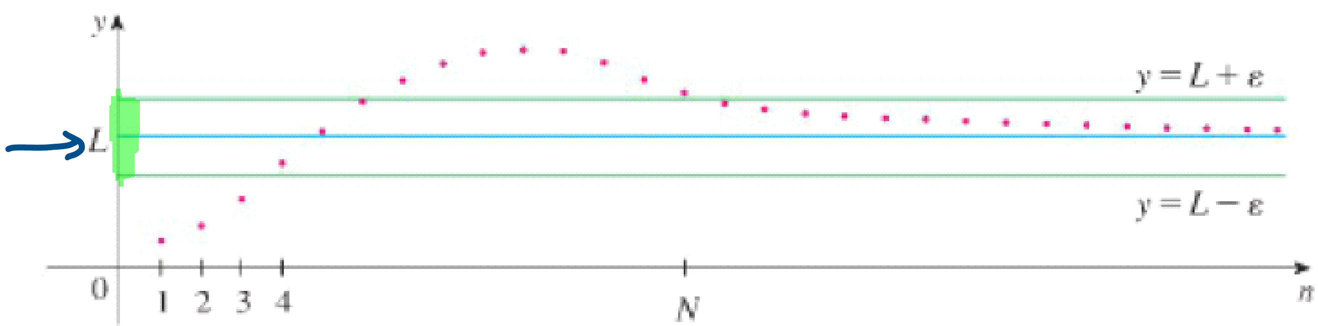
$a_2 = 1$

$a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$

Page 3: Limit def. of converge

Definition: A sequence $\{a_n\}$ is said to have the limit L, written $\boxed{\lim_{n\to\infty} a_n = L}$ or $a_n \to L$ as $n \to \infty$, if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say that the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

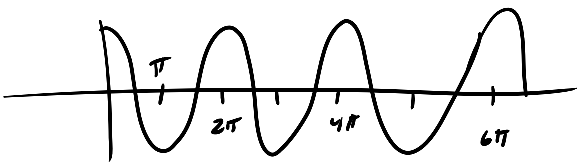
Definition: If $\{a_n\}$ is a sequence, then $\lim_{n\to\infty} a_n = L$ means that for every $\epsilon > 0$ there is a corresponding integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.



Example: Do these sequences converge or diverge.

A) $\{(-1)^n\}_{n=1}^{\infty}$ $= -1, 1, -1, 1, -1, 1, -1, 1 \ldots$ diverges.

B) $\{\cos(2n\pi)\}_{n=1}$ $= \underset{\cos(2\pi)}{1}, \underset{\cos(4\pi)}{1}, \underset{\cos(6\pi)}{1}, \ldots$ converges to 1



C) $\left\{\dfrac{3n}{n+2}\right\}_{n=5}^{\infty}$

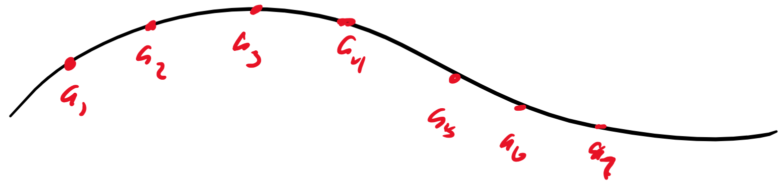
$$\lim_{n\to\infty} \frac{3n}{n+2} = \lim_{n\to\infty} \frac{3n}{n\left[1+\frac{2}{n}\right]} = \lim_{n\to\infty} \frac{3}{1+\frac{2}{n}} = \frac{3}{1+0} = 3$$

Seq. will converge to the # 3.

$f(x) = \dfrac{3x}{x+2}$ is the function containing a_n.

$$\lim_{x\to\infty} \frac{3x}{x+2} \overset{L'H}{=} = \lim_{x\to\infty} \frac{3}{1} = 3$$

THEOREM If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.



Example: Does the sequences converge or diverge? If it converges, give the value.

A) $\left\{ \frac{n^2}{\ln(3+e^n)} \right\}$

$$\frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{x^2}{\ln(3+e^x)} \overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{\frac{e^x}{3+e^x}} = \lim_{x\to\infty} \frac{2x\,(3+e^x)}{e^x}$$

$$= \lim_{x\to\infty} \frac{6x + 2xe^x}{e^x}$$

$$= \lim_{x\to\infty} \frac{6x}{e^x} + \frac{2xe^x}{e^x} = \lim_{x\to\infty} \frac{6x}{e^x} + 2x = 0 + \infty = \infty$$

The seq. will diverge

!!

$$\lim_{x\to\infty} \frac{6x}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{6}{e^x} = 0$$

B) $\left\{\frac{3n}{n+2}+\frac{n^2}{n^2+1}\right\}$

as $n \to \infty$ $a_n \to 3+1=4$

Seq. converges

$\lim_{n\to\infty} \frac{3n}{n+2} + \lim_{n\to\infty} \frac{n^2}{n^2+1} = \ldots = 3+1$

Limit Laws for Convergent Sequences: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n+b_n)=\lim_{n\to\infty}a_n+\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}(a_n-b_n)=\lim_{n\to\infty}a_n-\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}ca_n=c\lim_{n\to\infty}a_n$$

$$\lim_{n\to\infty}a_nb_n=\lim_{n\to\infty}a_n*\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim\limits_{n\to\infty}a_n}{\lim\limits_{n\to\infty}b_n},\quad if\ \lim_{n\to\infty}b_n\neq 0$$

Squeeze Theorem for Sequences: If $a_n \le b_n \le c_n$ for $n \ge n_o$ and $\lim_{n\to\infty}a_n=\lim_{n\to\infty}c_n=L$, then $\lim_{n\to\infty}b_n=L$

Example: Does the sequence a_n converge or diverge? If it converges, give the value.

A) $a_n = \dfrac{(-1)^n n^2}{n^2+1}$

alternating seq.

$a_n = (-1)^n b_n$

here $b_n = \dfrac{n^2}{n^2+1}$

as $n \to \infty$ $b_n \to 1$

$\lim_{n\to\infty} \dfrac{n^2}{n^2+1} \overset{L'H}{=} \cdots = 1$

Since $b_n \to 1$ as $n \to \infty$

we know $a_n = (-1)^n b_n$ will diverge

Alternating sequence.

$a_n = (-1)^n b_n$ will only conv if $b_n \to 0$ as $n \to \infty$

and will converge to the # zero.

B) $a_n = \dfrac{(-1)^n 3n}{n^2+5}$

alternating seq.

$$b_n = \frac{3n}{n^2+5}$$

as $n \to \infty$ $b_n \to 0$

Thus $a_n \to 0$ Converge..

C) $a_n = \dfrac{n!}{n^n}$

$0! = 1 \qquad 1! = 1$

$3! = 3 \cdot 2 \cdot 1$

$4! = 4 \cdot 3 \cdot 2 \cdot 1$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 5 \cdot 4!$

$= 5 \cdot 4 \cdot 3!$

all of these fractions are smaller than the # 1

$$0 < \underbrace{\frac{\boxed{n} \cdot (n-1)(n-2) \cdots\cdots 1}{\boxed{n} \cdot n \cdot n \cdot \cdots \cdot n}}_{n \text{ terms.}} = \underbrace{\boxed{\frac{(n-1)}{n}} \cdot \boxed{\frac{(n-2)}{n}} \cdots\cdots \boxed{\frac{2}{n}} \cdot \boxed{\frac{1}{n}}}_{n-1 \text{ terms } \le n} < \frac{1}{n}$$

$$b_n = 0 < \overbrace{\frac{n!}{n^n}}^{a_n} < \frac{1}{n} = c_n$$

as $n \to \infty$

$b_n \to 0$

$c_n \to 0$

by squeeze thm.

$a_n \to 0$ as $n \to \infty$

Example: Find the values of r so that $\{r^n\}$ converges. Determine what the series will converge to.

$r = 1$ conv. conv. to the # 1

$\left.\begin{array}{l} r > 1 \\ r < -1 \\ r = -1 \end{array}\right\}$ div. $r = -2$ $(-2)^n = (-1)^n 2^n$

$|r| < 1 \leftarrow \left[-1 < r < 1 \right]$ converge to the # zero. $r = \frac{1}{2}$ $\left(\frac{1}{2}\right)^n \longrightarrow 0$ as $n \to \infty$

$r = -\frac{1}{2}$ $(-1)^n \left(\frac{1}{2}\right)^n \longrightarrow 0$ as $n \to \infty$

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is $a_1 < a_2 < a_3 < \ldots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonoic** if it is either increasing or decreasing.

Example: Show that the sequence $a_n = \dfrac{n}{n^2+4}$ is a decreasing sequence.

$f(x) = \dfrac{x}{x^2+4}$ where a_n is in $f(x)$ i.e. $f(n) = a_n$ for the integers n.

$$f' = \frac{(x^2+4)(1) - x(2x)}{(x^2+4)^2} = \frac{x^2+4-2x^2}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$$

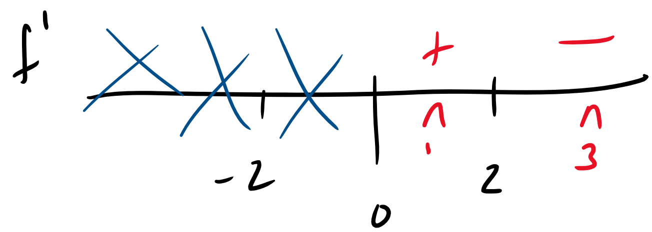
$f' = 0$

$$0 = \frac{4-x^2}{(x^2+4)^2}$$

$0 = 4 - x^2$

$x^2 = 4$

$x = \pm 2$



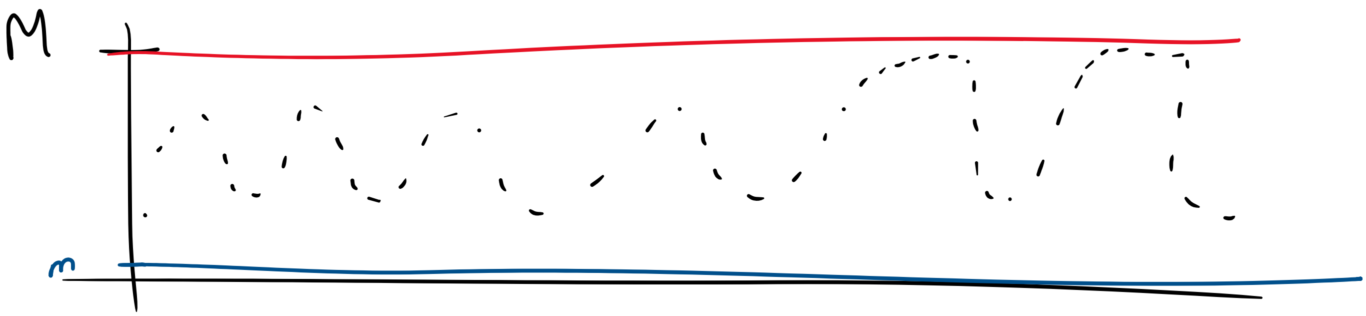
$f(x)$ is dec for $x > 2$

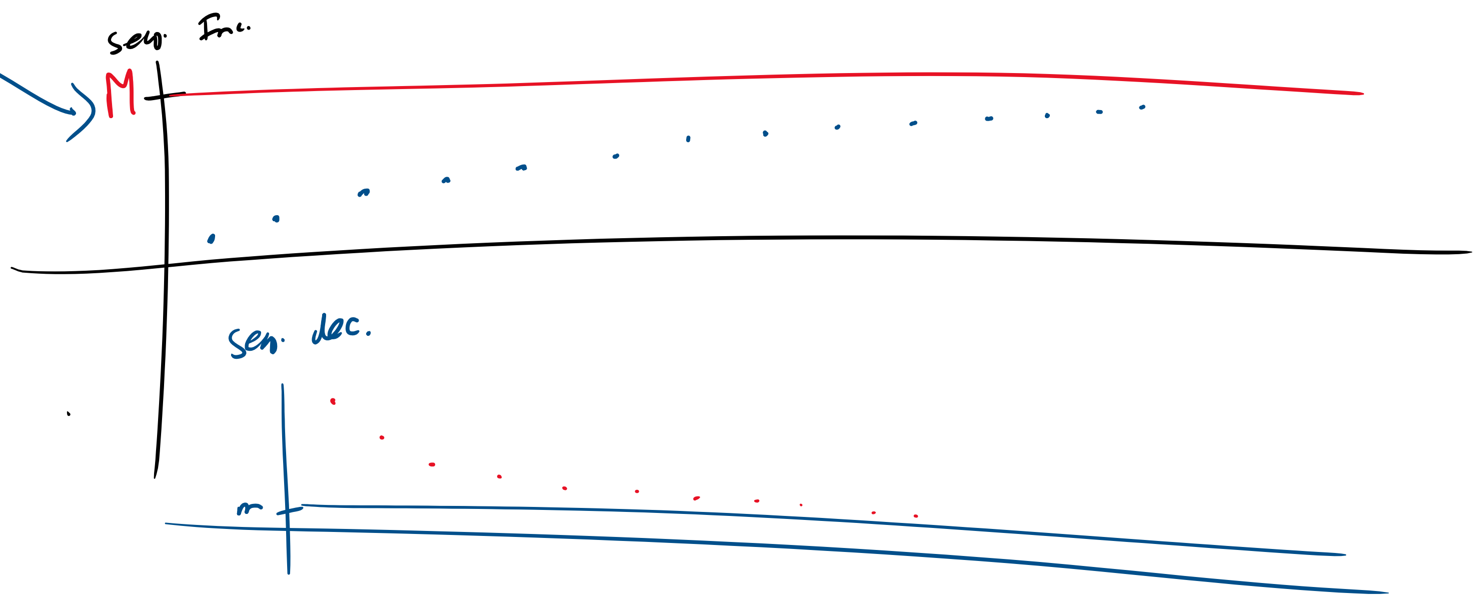
Inc. $0 < x < 2$

The seq. a_n is dec. for $n \geq 2$

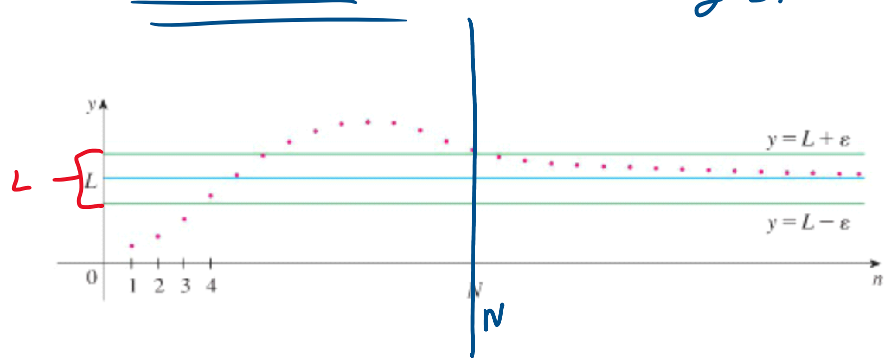
Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \le M$ for all $n \ge 1$. It is **bounded below** if there is a number m such that $m \le a_n$ for all $n \ge 1$. If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.





Question: If a sequence is convergent, is the sequence bounded? yes.



Example: You are given that the sequence given by $a_1 = \sqrt{5}$, $a_{n+1} = \sqrt{5 + a_n}$ is increasing and bounded above by 4.

$\rightarrow a_n = \sqrt{5 + a_{n-1}}$

Find $\lim_{n\to\infty} a_n$

say the seq. is conv.

$$\lim_{n\to\infty} a_n = L$$

if as $n \to \infty$ $a_n \to L$ $a_{n+1} \to L$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{5 + a_n}$$

$$L = \sqrt{5 + L}$$

$$L^2 = 5 + L$$

$$L^2 - L - 5 = 0$$

$$L = \frac{1 \pm \sqrt{1^2 - 4(1)(-5)}}{2(1)} = \frac{1 \pm \sqrt{21}}{2}$$

$$L = \frac{1 + \sqrt{21}}{2} > 0 \qquad L = \frac{1 - \sqrt{21}}{2} < 0$$

$a_1 = \sqrt{5}$ Increasing so answer is $\frac{1+\sqrt{21}}{2}$

Example: You are told that the sequence given by $a_1 = 1$, $a_{n+1} = 3a_n - 1$ is increasing. Does this sequence converge?

assume it conv.. i.e. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 3a_n - 1$

$L = 3L - 1$

$1 = 2L$

not possible → $\frac{1}{2} = L$

Thus the seq. does not conv.

Example: Assume that this sequence will converge. Give the exact value that it will converge to.

$a_1 = -1 \qquad a_{n+1} = \frac{1}{5}\left(a_n + \frac{44}{a_n}\right)$

as $n \to \infty$ $\quad a_n \to L$ and $a_{n+1} \to L$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{5}\left(a_n + \frac{44}{a_n}\right)$$

$$L = \frac{1}{5}\left(L + \frac{44}{L}\right)$$

$$5L = L + \frac{44}{L}$$

$$4L = \frac{44}{L}$$

$$4L^2 = 44$$

$$L^2 = 11$$

$$L = +\sqrt{11} \qquad L = -\sqrt{11}$$

formula shows all terms are neg if a_1 = a negative #.

Seq. conv. to $-\sqrt{11}$