Sections 5.3: The Fundamental Theorem of Calculus

Definition: The indefinite integral of f is used to indicate the process of finding the antiderivative of f. i.e. $\int f(x)\ dx = F(x) + C$

Example: Compute.

A) $\int 2x^5 + 7x + 4\ dx =$ $\frac{2x^6}{6} + \frac{7x^2}{2} + 4x + C$

$f'(x) = 2x^5 + 7x + 4$

$f(x) = \frac{2x^6}{6} + \frac{7x^2}{2} + 4x + C$

B) $\int 3x^2\ da =$ $3x^2 a + C$

a is the variable

C) $\int \frac{x^2 + 2x^5 + 7x^3 + 4}{4x^3}\,dx$ $= \int \frac{x^2}{4x^3} + \frac{2x^5}{4x^3} + \frac{7x^3}{4x^3} + \frac{4}{4x^3}\,dx$

$= \int \frac{1}{4x} + \frac{1}{2}x^2 + \frac{7}{4} + x^{-3}\,dx$

$\frac{1}{4} \cdot \frac{1}{x}$

$= \frac{1}{4}\ln|x| + \frac{1}{2}\frac{x^3}{3} + \frac{7}{4}x + \frac{x^{-2}}{-2} + C$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a), \qquad \text{where } F \text{ is any antiderivative of } f.$$

$=$

evaluate from $x=1$ to $x=5$

Example: Compute the following.

A) $\displaystyle\int_1^5 (3x^2 + 4x + 2)\,dx = \left(x^3 + 2x^2 + 2x + C\right)\Big|_1^5$

$= 5^3 + 2(5)^2 + 2(5) + C - \left[1^3 + 2(1)^2 + 2(1) + C\right]$

$= 125 + 50 + 10 + C - 1 - 2 - 2 - C$

$= 185 + C - 5 - C = 180$

B) $\int_0^4 3x + 8e^{4x}\ dx =$ $\left(\frac{3x^2}{2} + 8 \cdot \frac{1}{4}e^{4x}\right)\Big|_0^4$

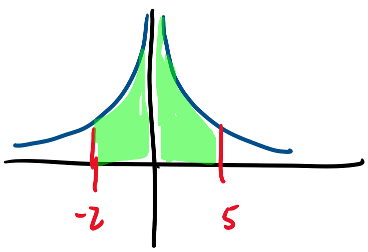
$= \left(\frac{3}{2}x^2 + 2e^{4x}\right)\Big|_0^4$

$= \frac{3}{2}(4)^2 + 2e^{4(4)} - \left[0 + 2e^0\right]$

$= 24 + 2e^{16} - 2 \quad = 22 + 2e^{16}$

C) $\int_{-2}^{5} \frac{1}{x^2}\,dx = \int_{-2}^{5} x^{-2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{-2}^{5}$

$= \left.-\frac{1}{x}\right|_{-2}^{5} = -\frac{1}{5} - \frac{-1}{-2} = -\frac{1}{5} - \frac{1}{2} = \frac{-7}{10}$



$\frac{1}{x^2}$ is not cont. on $[-2, 5]$ ie V.A. @ $x=0$

Can not use FTC for this problem.

D) $\int_0^3 |x^2 - 4| \, dx =$

$x^2 - 4 = 0$

$x = \pm 2$

$+ \quad - \quad +$

$-5 \quad -2 \quad 0 \quad 2 \quad 5$

$|x| \geq 2$

$$|x^2-4| = \begin{cases} x^2-4, & x \geq 2 \text{ or } x \leq -2 \\ -(x^2-4), & -2 < x < 2 \end{cases}$$

$$= \int_0^2 |x^2-4| \, dx + \int_2^3 |x^2-4| \, dx$$

$$= \int_0^2 -(x^2-4) \, dx + \int_2^3 x^2-4 \, dx$$

$$= \int_0^2 (-x^2+4) \, dx + \int_2^3 (x^2-4) \, dx = \left(-\frac{x^3}{3} + 4x\right)\Big|_0^2 + \left(\frac{x^3}{3} - 4x\right)\Big|_2^3$$

$$= \left(-\frac{8}{3} + 8\right) - (0) + (9 - 12) - \left(\frac{8}{3} - 8\right)$$

$$= -\frac{8}{3} + 8 - 3 - \frac{8}{3} + 8 = 13 - \frac{16}{3} = \frac{39}{3} - \frac{16}{3}$$

$$= \frac{23}{3}$$

$$\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)$$

E) $\int_4^9 \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 x - 1 - 1 + \frac{1}{x} \; dx = \int_4^9 x - 2 + \frac{1}{x} \; dx$

$$= \left(\frac{x^2}{2} - 2x + \ln|x|\right)\Big|_4^9$$

$$= \frac{81}{2} - 18 + \ln(9) - \left(8 - 8 + \ln(4)\right)$$

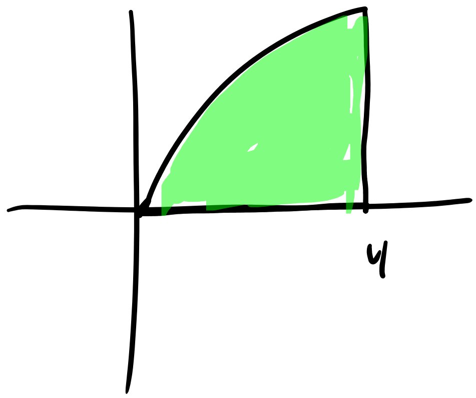
$$= \frac{81}{2} - 18 + \ln(9) - \ln(4)$$

$$= \frac{45}{2} + \ln\left(\frac{9}{4}\right)$$

$$\frac{81}{2} - \frac{36}{2}$$

$$\begin{array}{r} 81 \\ -36 \\ \hline 45 \end{array}$$

Example: Sketch the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 4$ and calculate its area.



$$\int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^4$$

$$= \frac{2}{3}(4)^{3/2} - 0$$

$$= \frac{2}{3}(8) = \frac{16}{3}$$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$\underbrace{g(x) = \int_a^x f(t)\,dt,}\qquad \text{with } a \le x \le b$$

is continuous on $[a, b]$ and is differentiable on (a, b), and $g'(x) = f(x)$

$$g(x) = \int_a^x f(t)\,dt = F(t)\Big|_a^x$$

where $\underbrace{F'(t) = f(t)}$

$$g(x) = F(x) - F(a)$$

$$g'(x) = F'(x) \cdot 1 - 0 = F'(x)$$

$$g'(x) = f(x)$$

Example: Find $g'(x)$.

were $F'(t) = f(t)$

$= t^2+1$

A) $g(x) = \int_a^x t^2 + 1 \, dt$ $= F(t) \Big|_a^x = F(x) - F(a)$

$g(x) = F(x) - F(a)$ $g'(x) = F'(x) \cdot 1 - 0$

$= x^2+1$

$g'(x) = x^2+1$

B) $g(x) = \int_4^{x^2} \tan^3(t) \, dt$ $= F(t) \Big|_4^{x^2}$ were $F'(t) = \tan^3(t)$

$g(x) = F(x^2) - F(4)$

$g' = F'(x^2) \cdot 2x - 0$

$g'(x) = \tan^3(x^2) \cdot 2x$

C) $g(x) = \int_{x^3}^{2} \ln(u)\, du = F(u)\Big|_{x^3}^{2}$ where $F'(u) = \ln(u)$

$$g(x) = -\int_{2}^{x^3} \ln(u)\, du$$

$$g(x) = F(2) - F(x^3)$$

$$g'(x) = 0 - F'(x^3) \cdot 3x^2$$

$$g'(x) = 0 - \ln(x^3) \cdot 3x^2$$

D) $g(x) = \int_{x^2}^{x^3+1} u^5 + 2\, du = F(u) \Big|_{x^2}^{x^3+1}$ where $F'(u) = u^5 + 2$

$g(x) = F(x^3+1) - F(x^2)$

$g'(x) = F'(x^3+1) \cdot 3x^2 - F'(x^2) \cdot 2x$

$= \left[(x^3+1)^5 + 2\right] \cdot 3x^2 - \left[(x^2)^5 + 2\right] \cdot 2x$

$g(x) = \int_{x^2}^{0} u^5+2\,du + \int_{0}^{x^3+1} u^5+2\,du$

$= -\int_{0}^{x^2} u^5+2\,du + \int_{0}^{x^3+1} u^5+2\,du$

Pg 13

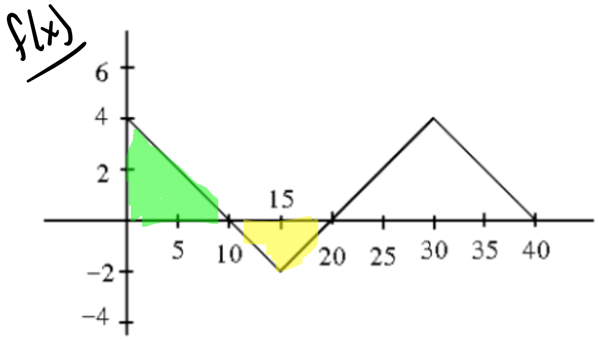
Example: Define $g(a)$ by $g(a) = \int_0^a f(x)\,dx$ where $f(x)$ is the graph given below.

1) Compute $g(10)$ and $g(20)$.

2) Find the intervals where $g(a)$ is increasing.

3) If possible, give the values of the absolute maximum and absolute minimum.

$f(x)$



$$g(10) = \underbrace{\int_0^{10} f(x)\,dx} = \tfrac{1}{2}(10)(4) = 20$$

$$g(20) = \underbrace{\int_0^{20} f(x)\,dx} = 20 - \tfrac{1}{2}(10)(2)$$

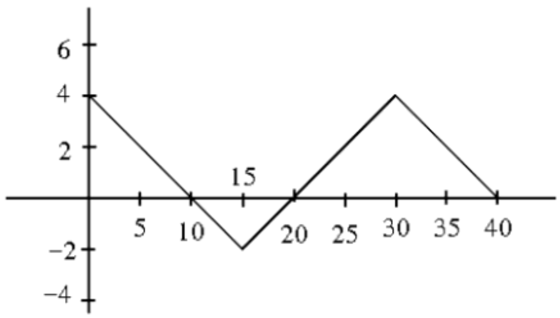
$$= 20 - 10 = 10$$

$$g(a) = \int_0^a f(x)\,dx$$

find where $g(a)$ is inc.

$$g'(a) = f(a)\cdot 1 - 0 = f(a)$$

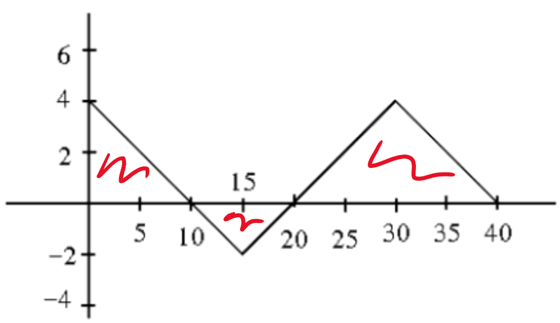
$f(x) \equiv g'(a)$



$g(a)$ Inc $(0,10) \cup (20,40)$

$g(a)$ dec $(10,20)$

C.V. $x=10$ & $x=20$



$g(0) = \int_0^0 f(x)dx = 0$ Abs min

$g(10) = 20$

$g(20) = 10$

$s(20) = 10$

$s(40) = \int_0^{40} f(x)\,dx$

$= 20 - 10 + \frac{1}{2}(20)4$

$= 20 - 10 + 40$

$= 50$ Abs max