Section 4.4: Indeterminate Forms and L'Hopital's Rule

$\frac{0}{0}$

$$\lim_{x\to 4} \frac{x^2-16}{x^2-5x+4} = \lim_{x\to 4} \frac{\cancel{(x-4)}(x+4)}{\cancel{(x-4)}(x-1)} = \lim_{x\to 4} \frac{x+4}{x-1} = \frac{8}{3}$$

$$\lim_{x\to 4} \frac{x^2-16}{x^2-5x+4} \overset{L'H}{=} \lim_{x\to 4} \frac{2x}{2x-5} = \frac{8}{8-5} = \frac{8}{3}$$

$$\lim_{x\to 0} \frac{2^x-1}{x} = $$

$\frac{0}{0}$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{2^x \ln(2)}{1} = 2^0 \ln(2) = \ln(2)$$

7 cases of indeterminate forms: $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 * \infty, 0^0, 1^\infty, \infty^0$

L'Hopital's Rule Suppose that $f(x)$ and $g(x)$ are differential functions and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). If

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} \overset{\text{L'H}}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit of the right side exists (or is ∞ or $-\infty$).

case $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example: Evaluate these limits:

A) $\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1-\cos(2x)} \overset{L'H}{=} \lim_{x\to 0} \frac{e^x - e^{-x}}{+\sin(2x)\cdot 2} \overset{L'H}{=} \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos(2x)\cdot 2^2} = \frac{1+1}{1\cdot 2^2} = \frac{2}{4} = \frac{1}{2}$

$\frac{0}{0}$ $\frac{0}{0}$

$$\frac{0}{0}$$

B) $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x}}{x^2} = \cancel{\overset{L'H}{} \lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{2x} \overset{L'H}{=} \lim\limits_{x\to 0} \dfrac{e^x + e^{-x}}{2} = \dfrac{1+1}{2} = \dfrac{2}{2} = 1}$

$$\frac{2}{0}$$

B) $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x}}{x^2} = +\infty$

$$\lim_{x\to 0} \frac{e^x + e^{-x}}{x} = \text{ONE}$$

Pg 5

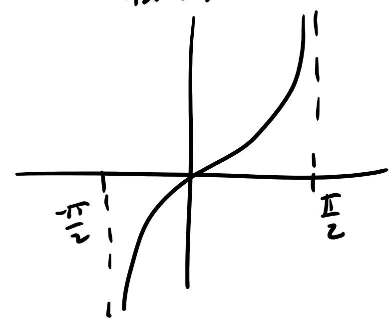
C) $\displaystyle \lim_{x\to \frac{\pi}{2}^-} \frac{4\tan(x)}{1+\sec(x)} \overset{\frac{\infty}{\infty}}{=}_{L'H} \lim_{x\to \frac{\pi}{2}^-} \frac{4\sec^2(x)}{\sec(x)\tan(x)}$

$\frac{\infty}{\infty}$

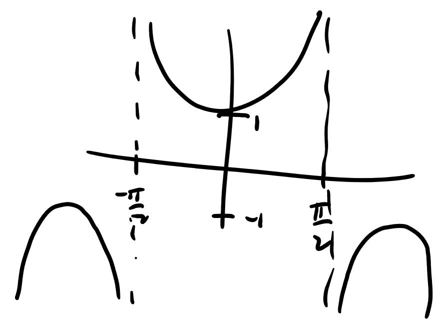
$$= \lim_{x\to \frac{\pi}{2}^-} \frac{4\sec(x)}{\tan(x)} = \lim_{x\to \frac{\pi}{2}^-} \frac{\frac{4}{\cos(x)}}{\frac{\sin(x)}{\cos(x)}}$$

$$= \lim_{x\to \frac{\pi}{2}^-} \frac{4}{\cos(x)} \cdot \frac{\cos(x)}{\sin(x)} = \lim_{x\to \frac{\pi}{2}^-} \frac{4}{\sin(x)} = \frac{4}{1} = 4$$

tan(x)



Sec(x)



case: $\infty - \infty$

$\frac{0}{0}$

Example: Evaluate these limits:

A) $\lim_{x \to \frac{\pi}{2}^-} \sec(x) - \tan(x) =$ $\lim_{x \to \frac{\pi}{2}^-} \frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)} = \lim_{x \to \frac{\pi}{2}^-} \frac{1-\sin(x)}{\cos(x)}$

$\infty - \infty$

$\overset{L'H}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{-\cos(x)}{-\sin(x)} = \frac{-0}{-1} = 0$

$$\frac{0}{0-0} = \frac{0}{0}$$

B) $\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1} \frac{x\ln(x) - (x-1)}{(x-1)\ln(x)} = \lim_{x\to 1} \frac{x\ln(x) - x + 1}{x\ln(x) - \ln(x)}$

$$\overset{L'H}{=} \lim_{x\to 1} \frac{1\ln(x) + x\cdot\frac{1}{x} - 1}{1\ln(x) + x\cdot\frac{1}{x} - \frac{1}{x}}$$

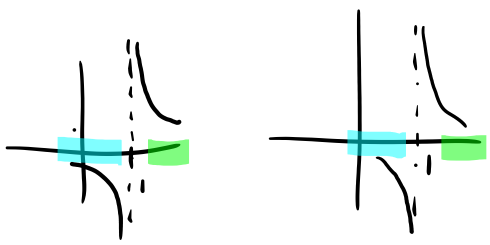
$$\frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\ln(x)}{\ln(x) + 1 - \frac{1}{x}}$$

$$\overset{L'H}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x} - \frac{-1}{x^2}} = \frac{1}{1 - -1} = \frac{1}{2}$$

$\frac{x}{x-1}$ $\qquad$ $\frac{1}{\ln(x)}$

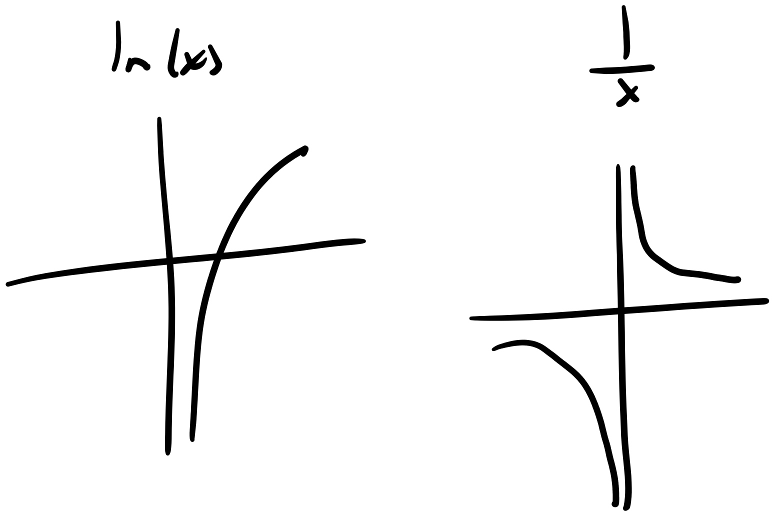
VA $x=1$



$\infty - \infty$

$-\infty - -\infty = -\infty + \infty$

C) $\lim_{x\to 0^+} \ln(x) - \frac{1}{x} =$ $-\infty - +\infty = -\infty - \infty = -\infty$



case: $0 * \infty$

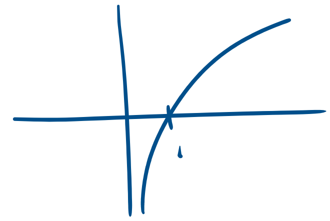
$$f(x)\cdot g(x) \longrightarrow \underset{\frac{\infty}{\infty}}{\frac{g(x)}{\frac{1}{f(x)}}} \quad \text{or} \quad \underset{\frac{0}{0}}{\frac{f(x)}{\frac{1}{g(x)}}}$$

$0 \quad \infty$

Example: Evaluate these limits:

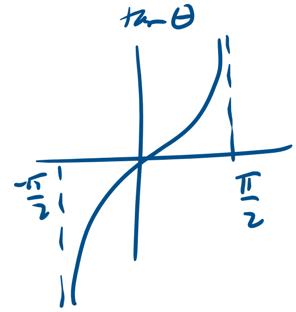
A) $\lim\limits_{x\to 1^+} \ln(x)\tan\left(\frac{\pi x}{2}\right) = \lim\limits_{x\to 1^+} \dfrac{\ln(x)}{\frac{1}{\tan\left(\frac{\pi x}{2}\right)}} = \lim\limits_{x\to 1^+} \dfrac{\ln(x)}{\cot\left(\frac{\pi x}{2}\right)}$

$0 \quad \infty$



$$\boxed{\frac{\tan\left(\frac{\pi x}{2}\right)}{\frac{1}{\ln(x)}} \quad \text{or} \quad \frac{\ln(x)}{\frac{1}{\tan\left(\frac{\pi x}{2}\right)}}}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 1^+} \frac{\frac{1}{x}}{-\csc^2\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}}$$



$$= \frac{1}{-\frac{\pi}{2}\csc^2\left(\frac{\pi}{2}\right)}$$

$$= \frac{1}{-\frac{\pi}{2}\cdot\frac{1}{\sin^2\left(\frac{\pi}{2}\right)}} = \frac{1}{-\frac{\pi}{2}} = \frac{-2}{\pi}$$

$0 \cdot \infty$

$$\frac{\sec(x)}{\frac{1}{2x-\pi}} \qquad \frac{2x-\pi}{\cos(x)}$$

B) $\lim\limits_{x \to \frac{\pi}{2}^-} (2x - \pi)\sec(x) =$

$\frac{0}{0}$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{2x-\pi}{\cos(x)} \overset{L'H}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{2}{-\sin(x)} = \frac{2}{-1} = \boxed{-2}$$

case: ∞^0, 1^∞, 0^0

$$\lim f(x)^{g(x)} = \lim e^{\ln f(x)^{g(x)}} = \lim e^{\boxed{g(x)\ln(f(x))}}$$

$$\boxed{\lim g(x)\ln(f(x))}$$

Example: Evaluate these limits:

A) $y = \lim\limits_{x\to 1^+} (2-x)^{\frac{4}{x-1}}$

$$\ln y = \lim_{x\to 1^+} \ln (2-x)^{\frac{4}{x-1}}$$

$$\ln(y) = \lim_{x\to 1^+} \frac{4\ln(2-x)}{x-1} \overset{L'H}{=} \lim_{x\to 1^+} \frac{4\cdot\frac{-1}{2-x}}{1} = \frac{\frac{-4}{2-1}}{1} = -4$$

$$\ln(y) = -4$$

$$y = e^{-4} = \lim_{x\to 1^+} (2-x)^{\frac{4}{x-1}}$$

1^{∞}

$$\frac{2\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} \quad \text{or} \quad \frac{2x}{\frac{1}{\ln\left(1+\frac{3}{x}\right)}}$$

B) $y = \lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{2x} =$

$\infty \cdot 0$

$$\ln y = \lim_{x\to\infty} \ln\left(1+\frac{3}{x}\right)^{2x} = \lim_{x\to\infty} 2x \ln\left(1+\frac{3}{x}\right)$$

$$= \lim_{x\to\infty} \frac{2\cdot\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{2\cdot\frac{-\frac{3}{x^2}}{1+\frac{3}{x}}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2\cdot -\frac{3}{x^2}}{1+\frac{3}{x}} \cdot \frac{-x^2}{1} = \lim_{x\to\infty} \frac{2\cdot 3}{1+\frac{3}{x}} = \frac{6}{1} = 6$$

$$\ln(y) = 6$$

$$y = e^{6} = \lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{2x}$$