Sections 4.1-4.3 Part 2: Increase, Decrease, Concavity, and Local Extrema

Definition: A **critical number** (**critical value**) is a number, c, in the domain of f such that $f'(c) = 0$ or $f'(c)$ DNE.

If f has a **local extrema** (local maxima or minima) at c then c is a critical value of $f(x)$.

Fermat's Theorem: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Example: Find the intervals where the function is increasing and the intervals where it is decreasing. Classify all critical values.

1) domain.

2) find the critical values.

3) make the 1st derivative sign chart

4) conclusion.

A) $y = x^3 + 3x^2 - 9x + 8$

1) domain is all Reals

2) $y' = 3x^2 + 6x - 9$

$y' =$ DNE none.

$y' = 0$ $0 = 3x^2 + 6x - 9$

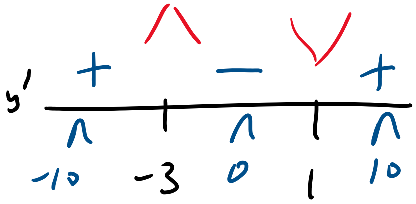
$0 = 3(x^2 + 2x - 3)$

$0 = 3(x+3)(x-1)$

$x = -3$ $x = 1$

C.V. $x = -3$, $x = 1$

3)



Inc $(-\infty, -3) \cup (1, \infty)$

Dec $(-3, 1)$

Local max @ $x = -3$ → Local max is $y(-3)$

Local min @ $x = 1$ → Local min is $y(1)$

B) $y = 3x^5 - 20x^3 + 20$

Domain: all Reals.

find C.V.

y' DNE: none.

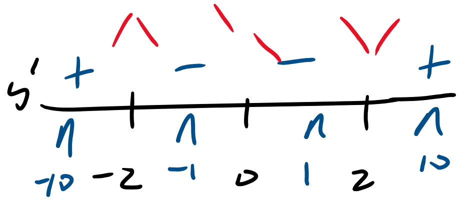
y'=0: $0 = 15x^2(x^2-4)$

$x = 0 \quad x = \pm 2$

C.V. $x=0$, $x=2$, $x=-2$

$y' = 15x^4 - 60x^2$

$y' = \underbrace{15x^2}_{\text{positive}}(x^2-4)$



Inc $(-\infty,-2)$, $(2,\infty)$

Dec $(-2,0)$ $(0,2)$ → $(-2,2)$ function is cont.

Local max @ $x=-2$

Local min @ $x=2$

@ $x=0$ is a neither.

C) $y = \frac{x^2+1}{x}$

Domain all Reals except $\overset{\text{V.A.}}{x=0}$

find C.N.

$\boxed{y' = \text{DNE}}$ x=0 not a C.N. since not in the domain.

$\boxed{y'=0}$

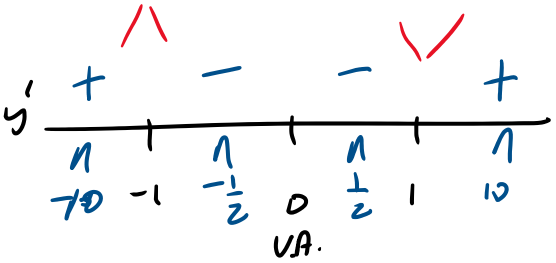
$$0 = \frac{x^2-1}{x^2}$$

$$0 = x^2-1 \rightarrow x = \pm 1$$

C.N. x = 1, x = -1

$$y' = \frac{x(2x) - (x^2+1)1}{x^2} = \frac{2x^2 - x^2 - 1}{x^2}$$

$$y' = \frac{x^2-1}{x^2}$$



Inc $(-\infty,-1)$, $(1,\infty)$

Dec $(-1,0)$, $(0,1)$

Local max @ x=-1

Local min @ x=1

D) $y = (x^2 - 16)^{2/3}$

Domain: all Real #s.

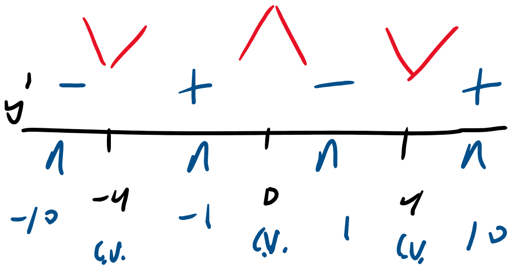
$$y' = \frac{2}{3}(x^2-16)^{-1/3} \cdot 2x = \frac{4x}{3(x^2-16)^{1/3}}$$

y' DNE: x=4, x=-4

These are C.V.

y'=0: x=0

Also a C.V.



f(x) is Inc (-4,0) (4,∞)

Dec (-∞,-4) (0,4)

local min @ x=-4

local min @ x=4

Local max at x=0

local max is $(-16)^{2/3}$

E) $y = x\ln(x)$ Domain is $x > 0$

$y' = 1\ln(x) + x \cdot \frac{1}{x}$

$y' = \ln(x) + 1$

y' DNE for $x \le 0$

not in the domain so not a C.V.

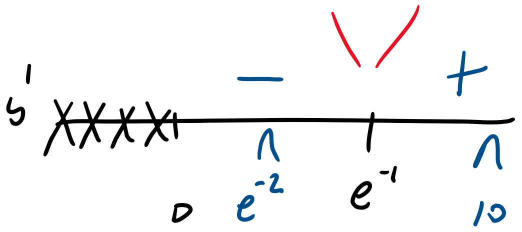
$y' = 0$

$0 = \ln(x) + 1$

$-1 = \ln(x)$

$x = e^{-1}$

C.V.



Inc (e^{-1}, ∞)

Dec $(0, e^{-1})$

Local min @ $x = e^{-1}$

F) $y' = \frac{(x-4)^3(x+2)^2}{(x-1)}$ with the domain of y being all real numbers except $x = 1$.

y' DNE @ $x=1$

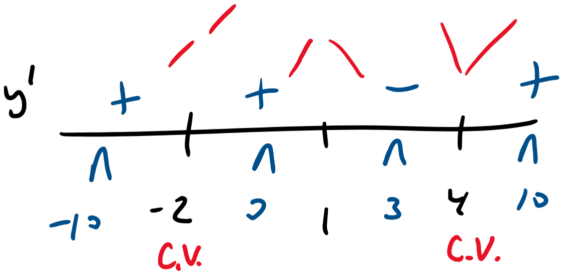
not a C.V. since not in the domain.

$y' = 0$

$x=4$

$x=-2$

C.V.



y is Inc $(-\infty,-2)$ $(-2,1)$ $(4,\infty)$

Dec $(1,4)$

Local min @ $x=4$

No local max

@ $x=-2$ is a neither.

Definition: $x = c$ is a possible inflection value (piv) provided that $x = c$ is in the domain of $f(x)$ and $f''(c) = 0$ or $f''(c)$ DNE.

Example: Find the intervals where the function is concave up and the intervals where it is concave down. Find the x-coordinate of the inflection points.

$y = x^5 - 5x^4 + 10x + 5$ Domain all Reals.

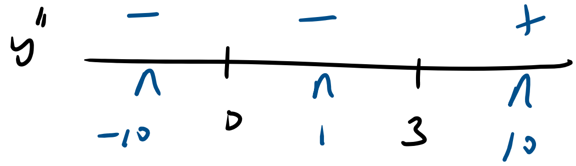
$y' = 5x^4 - 20x^3 + 10$

$y'' = 20x^3 - 60x^2$

$y'' = 20x^2(x-3)$

y'' DNE none.

$y''=0$ $x=0$ $x=3$



f(x) is C.U $(3,\infty)$

f(x) is C.D $(-\infty,0)$ $(0,3)$

Inflection pt at $x=3$

Example: Find the values of a and b so that $f(x) = ax^2 - b\ln(x)$ will have an inflection point at $(1, 5)$

$f''(1) = 0$ or $f''(1)$ DNE

$f(1) = 5$

domain is $x > 0$

$$f' = 2Ax - \frac{B}{x} = 2Ax - Bx^{-1}$$

$$f'' = 2A + Bx^{-2} = 2A + \frac{B}{x^2}$$

f'' DNE at $x = 0$

$$f''(1) = 0$$

$$2A + \frac{B}{1^2} = 0$$

$$2A + B = 0$$

$$5 = A(1)^2 - B\ln(1)$$

$$\boxed{5 = A}$$

$$B = -2A$$

$$\boxed{B = -10}$$

Example: The domain of the function $f(x)$ is all real numbers except $x = -5$. Use this information as well as f' and f'' to sketch a possible graph for $f(x)$.

$$f'(x) = \frac{-3x+7}{(x+5)^3} \qquad f''(x) = \frac{6(x-6)}{(x+5)^4}$$

$f'(x)$ info

$f'(x)$ DNE @ $x=-5$
not A C.V.

$f'(x)=0$

$0 = -3x+7$
$3x = 7$
$x = \frac{7}{3}$ C.V.

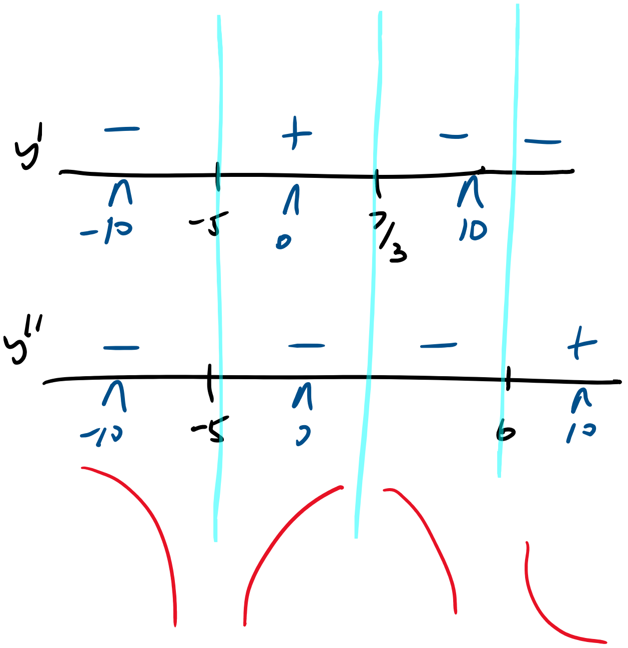
Inc $(-5, 7/3)$
Dec $(-\infty, -5)$ $(7/3, \infty)$

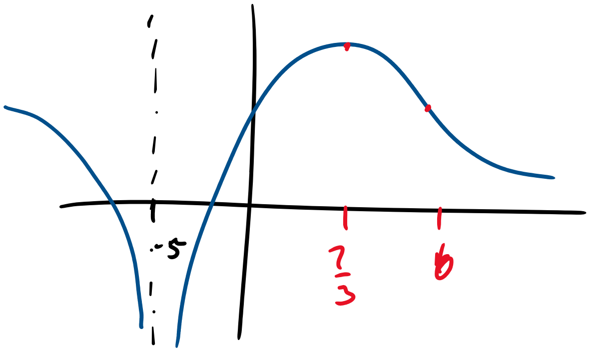
Local max @ $x = \frac{7}{3}$

$$f''(x) = \frac{6(x-6)}{(x+5)^4}$$

$f''(x)$ DNE
@ $x=-5$
not a p.i.v.

$f''(x)=0$
@ $x=6$





Second Derivative Test: Suppose that f'' is continuous near the critical value c.

(a) If $f''(c) > 0$ then $f(x)$ has a Local min at $x = c$. ∪

(b) If $f''(c) < 0$ then $f(x)$ has a local max at $x = c$. ∩

(c) If $f''(c) = 0$ then no conclusion can be made.

Example: Suppose that f has critical values of $x = 0$, $x = 2$, and $x = -2$. If $f''(x) = 60x^3 - 120x$, what conclusion can be drawn about the critical values?

$f''(x) = 60x(x^2-2)$

$f''(2) = 60(2)(4-2)$
$= 120(2)$
$= 240 > 0$ ∪

Local min at $x=2$

$f''(-2) = 60(-2)(4-2)$
$= -120(2)$
$= -240 < 0$ ∩

Local max at $x=-2$

$f''(0) = 0$

no Idea.

Example: What conclusion can be made if you know that $g''(5) = 7$?

Concave up at $x=5$

no idea if $x=5$ is a C.V.
So can not say it
it is a Local min.