Section 3.9: Related Rates

Example: Find $\frac{dx}{dt}$ when $y = 3$ and $\frac{dy}{dt} = 5$

$4x^3 - 5y^2 = -13$

if $y = 3$

$$4x^3 - 5(3)^2 = -13$$
$$4x^3 - 45 = -13$$
$$4x^3 = 32$$
$$x^3 = 8$$
$$x = 2$$

$$12x^2 \frac{dx}{dt} - 10y \frac{dy}{dt} = 0$$

$$12(2)^2 \frac{dx}{dt} - 10(3)(5) = 0$$

$$48 \frac{dx}{dt} - 150 = 0$$

$$48 \frac{dx}{dt} = 150$$

$$\frac{dx}{dt} = \frac{150}{48}$$

Example: At noon ship A leaves a port traveling North at 35km/hr. Ship B leaves the same port traveling East at 1pm at 25 km/hr. At what rate is the distance between them changing at 3pm?

$\frac{dx}{dt} = 35 \text{ km/hr}$

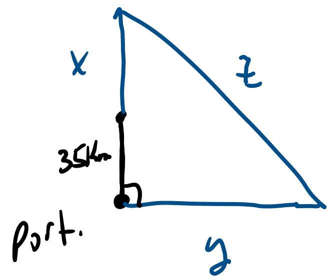
$\frac{dy}{dt} = 25 \text{ km/hr}$

find $\frac{dz}{dt}$

at 3pm

$y = 50 \text{ km}$

$x = 70 \text{ km}$



$$(x+35)^2 + y^2 = z^2$$

$$2(x+35)\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

find z

$$(70+35)^2 + (50)^2 = z^2$$

$$z = 116.257 \text{ km}$$

$$\frac{2(70+35)(35) + 2(50)(25)}{2(116.257)} = \frac{dz}{dt}$$

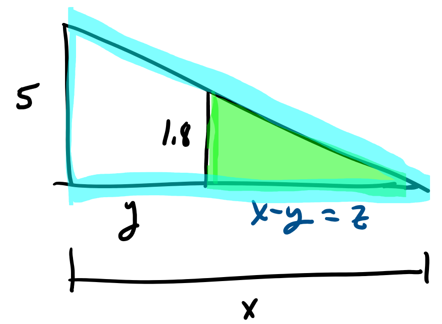
$$\frac{dz}{dt} = 42.35 \text{ km/hr.}$$

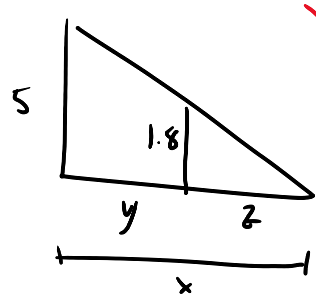
Example: A person 1.8 meters tall is walking away from a 5meter high lamppost at a rate of 2m/sec. At what rate is the end of the person's shadow moving away from the lamppost when the person in 6 meters from the lamppost?

$$\frac{dy}{dt} = 2m/sec.$$

find $\frac{dx}{dt}$

when $y = 6m$.





$y + z = x$

$$\frac{z}{1.8} = \frac{x}{5}$$

Similar triangles

$$\frac{x-y}{1.8} = \frac{x}{5}$$

$$5(x-y) = 1.8x$$

$$5x - 5y = 1.8x$$

$$3.2x = 5y$$

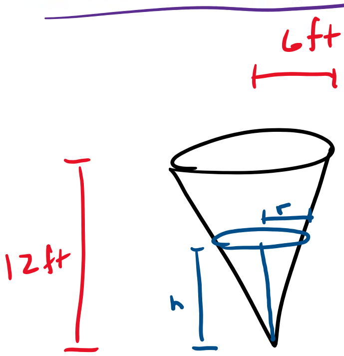
$$3.2\frac{dx}{dt} = 5\frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{5}{3.2}\frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{5(2)}{3.2} = \frac{10}{3.2} \text{ m/sec}$$

$$= 3.125 \text{ m/sec}$$

Example: A water tank has the shape of an inverted right circular cone of altitude 12 ft and base radius of 6ft. If water is being pumped into the tank at a rate of 10 gal/min (approximately 1.337 cubic feet per min) approximate the rate at which the water level is rising when the water is 3 feet deep.



$$\frac{dV}{dt} = 10\,gal/min = 1.337\ ft^3/min$$

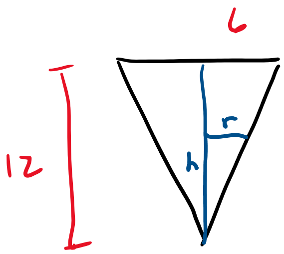
find $\frac{dh}{dt}$ when $h = 3$ ft.

$$\underbrace{V = \frac{1}{3}\pi r^2 h}$$

$$\boxed{\frac{dV}{dt} = \frac{2\pi}{3} r h \frac{dr}{dt} + \frac{\pi}{3} r^2 \frac{dh}{dt}}$$

$r = \frac{1}{2}h$

$\frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$



$\frac{r}{h} = \frac{6}{12}$

$\frac{r}{h} = \frac{1}{2}$

$r = \frac{1}{2}h$

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$

$$V = \frac{\pi}{3} \cdot \frac{h^2}{4} h$$

$$V = \frac{\pi}{12} h^3$$

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$

$$1.337 = \frac{\pi}{4}(3)^2 \cdot \frac{dh}{dt}$$

$$\frac{(1.337)(4)}{9\pi}\ ft/min = \frac{dh}{dt}$$

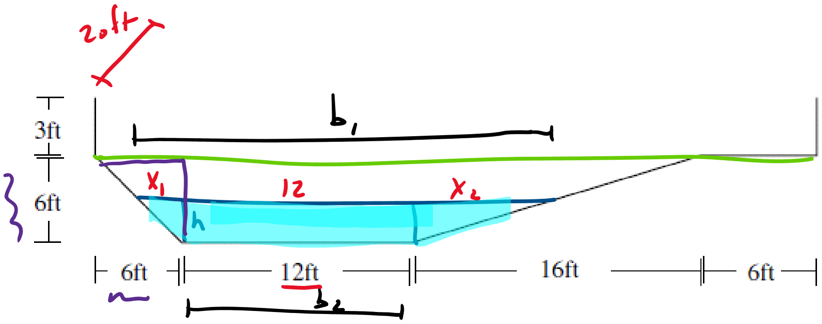
$$\frac{dh}{dt} = .1891\ ft/min$$

Example: A swimming pool is 20ft wide, 40ft long, 3ft deep at the shallow end and 9 ft deep at its deepest point. See the figure for a cross section. If the pool is being filled at a rate of 180 cubic feet per min, how fast is the water level rising when the depth at the deepest point is 4 ft?

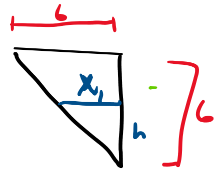
$$\frac{dv}{dt} = 180 \text{ ft}^3/\text{min}$$

find $\frac{dh}{dt}$ when

$h = 4$ ft



$$V = \frac{1}{2}(b_1 + b_2)h \cdot 20$$



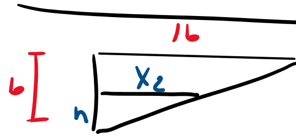
$$\frac{x_1}{h} = \frac{6}{6}$$

$$\frac{x_1}{h} = 1$$

$$x_1 = h$$

$$V = \left(\frac{1}{2}x_1 h + 12h + \frac{1}{2}x_2 h\right)20$$

$$= \frac{1}{2}(x_1 + 24 + x_2)\cdot h \cdot 20$$



$$\frac{x_2}{h} = \frac{16}{6}$$

$$\frac{x_2}{h} = \frac{8}{3}$$

$$x_2 = \frac{8}{3}h$$

$$V = 10\left(h + 24 + \frac{8}{3}h\right)\cdot h$$

$$= 10\left(\frac{11}{3}h + 24\right)h$$

$$V = \frac{110}{3}h^2 + 240h$$

$$\frac{dv}{dt} = \frac{220}{3}h\frac{dh}{dt} + 240\frac{dh}{dt}$$

$$\frac{dv}{dt} = \left(\frac{220}{3}h + 240\right)\frac{dh}{dt}$$

$$180 = \left(\frac{220}{3}\cdot 4 + 240\right)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{180}{\frac{220}{3}\cdot 4 + 240} \text{ ft/min} = .3375 \text{ ft/min}$$

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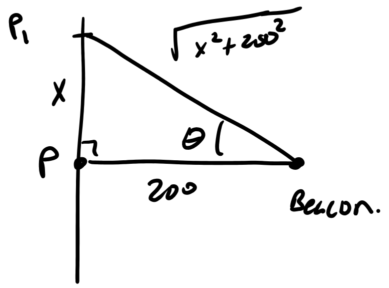
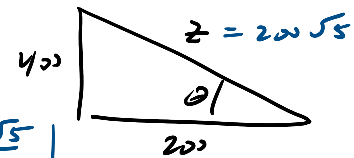
$$\frac{4\,Rev}{min}\cdot\frac{2\pi\,Rad.}{1\,Rev} = 8\pi\ Rad/min.$$

Example: A revolving beacon in a lighthouse makes one revolution every 15 seconds. The beacon is 200ft from the nearest point P on a straight shoreline. Find the rate at which a ray from the light moves along the shore at a point 400 ft from P.

$$\frac{4\,Rev}{min} \to \frac{d\theta}{dt} = \frac{4(2\pi)\,Rad.}{min}$$



find $\frac{dx}{dt}$ when $x = 400$



$$\tan\theta = \frac{x}{200}$$

$$\sec^2\theta\ \frac{d\theta}{dt} = \frac{1}{200}\ \frac{dx}{dt}$$

$$(\sqrt{5})^2(8\pi) = \frac{1}{200}\frac{dx}{dt}$$

$$\frac{dx}{dt} = 5(8\pi)\cdot 200\ ft/min.$$

$$\sec\theta = \frac{200\sqrt{5}}{200}$$

$$\sec\theta = \sqrt{5}$$

$$\tan\theta = \frac{400}{200}$$

$$\tan\theta = 2$$

$$z = \sqrt{(400)^2 + (200)^2}$$

$$z = \sqrt{160000 + 40000}$$

$$z = \sqrt{200000} = \sqrt{20\cdot 100\cdot 100}$$

$$= 100\sqrt{20}$$

$$= 200\sqrt{5}$$

Alternate method

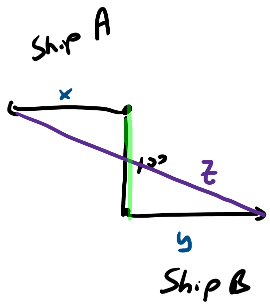
$$\tan\theta = \frac{x}{200}$$

$$\theta = \arctan\left(\frac{x}{200}\right)$$

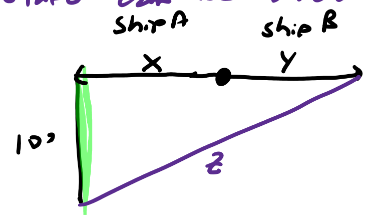
$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{x}{200}\right)^2}\cdot\frac{1}{200}\frac{dx}{dt}$$

Example: At noon, ship A is 100km north of ship B. Ship A travels west at 35km/hr and ship B is traveling east at 25km/hr. Find how the distance between the ships is changing at 3pm.

Actual picture.



picture can be drawn as



$$\frac{dx}{dt} = 35$$

$$\frac{dy}{dt} = 25$$

find $\frac{dz}{dt}$ at 3pm

$$(x+y)^2 + (100)^2 = z^2$$

at 3pm

$x = 105$

$y = 75$

$$(105+75)^2 + (100)^2 = z^2$$

$\vdots$

$$z = 205.913$$

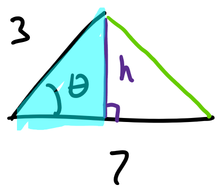
$$2(x+y)\cdot\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2z\frac{dz}{dt}$$

$$2(105+75)\cdot(35+25) = 2(205.913)\frac{dz}{dt}$$

$\vdots$

$$\frac{dz}{dt} = 52.449 \text{ Km/hr}$$

Example: Two sides of a triangle have fixed lengths of 3ft and 7ft. The angle between these sides is decreasing at a rate of 0.05 rad/sec. Find the rate at which the area of the triangle is changing when the angle between the fixed sides is 1 radian.



$\frac{d\theta}{dt} = -.05$ Rad/sec.

find $\frac{dA}{dt}$ when $\theta = 1$ Rad.

$A = \frac{1}{2} bh$

$= \frac{1}{2}(7)\, h$

$\sin\theta = \frac{h}{3}$

$3\sin\theta = h$

$A = \frac{1}{2}(7)\cdot 3\sin\theta$

$A = \frac{21}{2}\sin\theta$

$\frac{dA}{dt} = \frac{21}{2}\cos\theta \, \frac{d\theta}{dt}$

$\frac{dA}{dt} = \frac{21}{2}\cos(1)\,(-.05)$ ft²/sec

$= -.2837$ ft²/sec.