Section 3.6: Derivatives of Logarithmic Functions

Derivative Formulas:

$y = \ln(x)$

$e^y = x$

$e^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

$y = \ln(x)$

$y' = \frac{1}{x}$

$y = \ln f(x)$

$y' = \frac{1}{f(x)} \cdot f'(x)$

$y' = \frac{f'(x)}{f(x)}$

$y = \log_a(x) = \frac{\ln(x)}{\ln(a)} = \frac{1}{\ln(a)} \cdot \ln(x)$

$y' = \frac{1}{\ln(a)} \cdot \frac{1}{x}$

$y = \log_a f(x)$

$y' = \frac{f'(x)}{f(x) \ln a}$

Example: Find the derivative of these functions.

A) $y = \ln(5x^2 - 1)$

$$y' = \frac{1}{5x^2-1} \cdot 10x = \frac{10x}{5x^2-1}$$

B) $y = \ln(e^{x^2} + e^{-5x})$

$$y' = \frac{2xe^{x^2} - 5e^{-5x}}{e^{x^2} + e^{-5x}}$$

C) $y = \log_4(x^7 + 3x)$

$$y' = \frac{7x^6+3}{(x^7+3x)\ln(4)}$$

D) $y = e^{\log(x^2+1)}$

$$y' = e^{\log(x^2+1)} \cdot \frac{2x}{(x^2+1)\ln(10)}$$

E) $y = 5x\log(\underbrace{\cot(x^2)})$

$$y' = 5\log(\cot(x^2)) + 5x\,\frac{-\csc^2(x^2)\cdot 2x}{\cot(x^2)\ln(10)}$$

F) $y = \log_5\left[(x+4)^3(x^4+1)^2\right]$

$$y' = \frac{3(x+4)^2 \cdot 1\,(x^4+1)^2 + (x+4)^3 \cdot 2(x^4+1)^1 \cdot 4x^3}{(x+4)^3\,(x^4+1)^2\,\ln(5)}$$

$$= \frac{(x+4)^2(x^4+1)^1\left[3(x^4+1) + (x+4)\,8x^3\right]}{(x+4)^3(x^4+1)^2\,\ln(5)}$$

$$= \frac{3x^4 + 3 + 8x^4 + 32x^3}{(x+4)(x^4+1)\ln(5)} = \frac{11x^4 + 32x^3 + 3}{(x+4)(x^4+1)\ln(5)} = y'$$

F) $y = \log_5\left[(x+4)^3(x^4+1)^2\right]$

$\ln(AB) = \ln(A) + \ln(B)$

$\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$

$\ln(A^x) = x\ln(A)$

$$= \log_5(x+4)^3 + \log_5(x^4+1)^2$$

$$y = 3\log_5(x+4) + 2\log_5(x^4+1)$$

$$y' = 3 \cdot \frac{1}{(x+4)\ln 5} + 2 \cdot \frac{4x^3}{(x^4+1)\ln(5)}$$

G) $y = \ln\left(\frac{x^5+7}{\sqrt[5]{x^4+2}}\right) = \ln(x^5+7) - \ln(x^4+2)^{1/5}$

$y = \ln(x^5+7) - \frac{1}{5}\ln(x^4+2)$

$y' = \frac{5x^4}{x^5+7} - \frac{1}{5} \cdot \frac{4x^3}{x^4+2}$

x^2 $\quad$ $e^{\cos x}$

Logarithmic Differentiation

Example: Find the derivative.

A) $y = x^{\cos(x)}$

$$\ln y = \ln\left(x^{\cos(x)}\right)$$

$$\ln y = \cos(x)\ln(x)$$

$$\frac{y'}{y} = -\sin(x)\ln(x) + \cos(x)\cdot\frac{1}{x}$$

$$y' = y\left[-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right]$$

$$y' = x^{\cos x}\left[-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right]$$

B) $y = (x^3+7)^{e^{2x}}$

$$\ln y = \ln (x^3+7)^{e^{2x}}$$

$$\ln y = e^{2x} \ln(x^3+7)$$

$$\frac{y'}{y} = 2e^{2x} \ln(x^3+7) + e^{2x} \cdot \frac{3x^2}{x^3+7}$$

$$y' = (x^3+7)^{e^{2x}} \left[2e^{2x} \ln(x^3+7) + \frac{3x^2 e^{2x}}{x^3+7} \right]$$

Example: Find the derivative.

$$y = \frac{(x^3+1)^4(x^5+2)^8}{(6x^5+7)^5}$$

$$\ln y = \ln \frac{(x^3+1)^4(x^5+2)^8}{(6x^5+7)^5}$$

$$= \ln\left((x^3+1)^4(x^5+2)^8\right) - \ln(6x^5+7)^5$$

$$= \ln(x^3+1)^4 + \ln(x^5+2)^8 - \ln(6x^5+7)^5$$

$$\ln y = 4\ln(x^3+1) + 8\ln(x^5+2) - 5\ln(6x^5+7)$$

$$\frac{y'}{y} = 4\cdot\frac{3x^2}{x^3+1} + 8\cdot\frac{5x^4}{x^5+2} - 5\,\frac{30x^4}{6x^5+7}$$

$$y' = \frac{(x^3+1)^4(x^5+2)^8}{(6x^5+7)^5}\left[4\cdot\frac{3x^2}{x^3+1} + 8\cdot\frac{5x^4}{x^5+2} - 5\,\frac{30x^4}{6x^5+7}\right]$$

$$y = \frac{\sqrt{x^3+1}}{x^2+1}$$

$$y' = \frac{(x^2+1)\cdot \frac{1}{2}(x^3+1)^{-1/2}\cdot 3x^2 - \sqrt{x^3+1}\cdot 2x}{(x^2+1)^2}$$

$$\frac{1}{\sqrt{x^3+1}} \cdot \sqrt{x^3+1} = 1$$

$$y' = \frac{(x^2+1)\cdot \frac{1}{2}(x^3+1)^{-1/2}\cdot 3x^2 - \sqrt{x^3+1}\cdot 2x}{(x^2+1)^2} \cdot \frac{2\sqrt{x^3+1}}{2\sqrt{x^3+1}}$$

$$= \frac{(x^2+1)\cdot 3x^2 - 4x(x^3+1)}{2(x^2+1)^2\sqrt{x^3+1}}$$