## Section 3.5: Implicit Differentiation

Example: Examine the derivative of $x^2 + y^2 = 16$

$$y^2 = 16 - x^2$$

$$y = \pm\sqrt{16 - x^2}$$

$$y = \sqrt{16 - x^2} \qquad y = -\sqrt{16 - x^2}$$

The variable $y$ is a function of $x$.

Find the rate of change of $y$ with respect to $x$ i.e. $\frac{dy}{dx}$

$$2x \underbrace{\frac{dx}{dx}}_{1} + 2y\frac{dy}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

Example: Compute $\frac{dy}{dx}$. $x^3 + 2y^3 = (4x)(y)$

$$3x^2 + 6y^2 \frac{dy}{dx} = 4y + 4x \cdot 1 \frac{dy}{dx}$$

$$6y^2 \frac{dy}{dx} - 4x \frac{dy}{dx} = 4y - 3x^2$$

$$(6y^2 - 4x) \frac{dy}{dx} = 4y - 3x^2$$

$$\frac{dy}{dx} = \frac{4y - 3x^2}{6y^2 - 4x} = \frac{3x^2 - 4y}{4x - 6y^2}$$

Example: Compute $\frac{dy}{dx}$. $\tan(x^3) - 4xy^2 + e^{x^2} = \cos(3y)$

$$\tan(x^3) + e^{x^2} = (4x)(y^2) + \cos(3y)$$

$$\sec^2(x^3) \cdot 3x^2 + 2xe^{x^2} = 4y^2 + 4x \cdot 2y\frac{dy}{dx} - \sin(3y) \cdot 3\frac{dy}{dx}$$

$$\sec^2(x^3)\,3x^2 + 2xe^{x^2} - 4y^2 = \left[8xy - 3\sin(3y)\right]\frac{dy}{dx}$$

$$\frac{\sec^2(x^3)\,3x^2 + 2xe^{x^2} - 4y^2}{8xy - 3\sin(3y)} = \frac{dy}{dx}$$

Example: Compute $\frac{dy}{dx}$ and $\left.\frac{dy}{dx}\right|_{(-1,1)}$.

$x = \frac{3-y^2}{x-y}$

$x=-1$
$y=1$

$$x(x-y) = 3-y^2$$

$$x^2 - xy = 3 - y^2$$

$$x^2 = (x)(y) + 3 - y^2$$

$$2x = 1y + x \cdot 1 \frac{dy}{dx} - 2y\frac{dy}{dx}$$

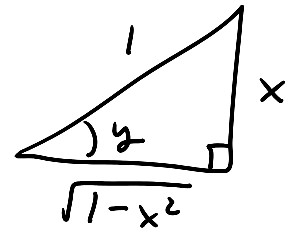
$$2x - y = (x-2y)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{2x-y}{x-2y}}$$

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{2(-1)-1}{-1-2(1)} = \frac{-3}{-3} = 1$$

Example: Compute $\frac{dy}{dx}$. $\underbrace{y = \sin^{-1}(x)}$

$$\sin(y) = x$$




$$\cos(y) = \frac{\sqrt{1-x^2}}{1}$$

$$\cos(y) \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

**Derivatives of Inverse Trigonometric Functions**

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\csc^{-1}(x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2}$$

$y = \arcsin^{-1}(\square)$

$$y' = \frac{1}{\sqrt{1-\square^2}} \frac{d}{dx}\square$$

Example: Find the derivative.

A) $y = \left(\sin^{-1}(2x)\right)^2$

$$y' = 2\left(\sin^{-1}(2x)\right)^1 \cdot \frac{d}{dx}\sin^{-1}(2x)$$

$$= 2\left(\sin^{-1}(2x)\right)^1 \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 = \frac{4\sin^{-1}(2x)}{\sqrt{1-4x^2}}$$

B) $y = \cos^{-1}(4x^2)$

$$y' = \frac{-1}{\sqrt{1-(4x^2)^2}} \cdot 8x = \frac{-8x}{\sqrt{1-16x^4}}$$

C) $y = \arctan(\sin(4x))$

$$y' = \frac{1}{1 + (\sin(4x))^2} \cdot \cos(4x) \cdot 4 = \frac{4\cos(4x)}{1 + \sin^2(4x)}$$

D) $y = \underbrace{x^3}_{f} \underbrace{\sec^{-1}(5x)}_{g}$

$f = x^3$

$f' = 3x^2$

$g = \sec^{-1}(5x)$

$g' = \dfrac{1}{5x\sqrt{(5x)^2-1}} \cdot 5$

$$y' = 3x^2 \cdot \sec^{-1}(5x) + x^3 \cdot \frac{1}{5x\sqrt{(5x)^2-1}} \cdot 5$$

$y = \sec^{-1}(x)$

$y' = \dfrac{1}{x\sqrt{x^2-1}}$

$y = \sec^{-1}(\square)$

$y' = \dfrac{1}{\square\sqrt{\square^2-1}} \dfrac{d}{dx}\square$