Section 3.1: Derivatives of Polynomials and Exponential Functions

Theorem: If f is a constant function, $f(x) = c$, then $f'(x) = 0$.

Theorem: If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$

Theorem: Let c be a constant and let $f'(x)$ and $g'(x)$ exists, then

a) if $y = cf(x)$, then $y' = cf'(x)$

b) if $y = f(x) \pm g(x)$, then $y' = f'(x) \pm g'(x)$

$y = \left(\frac{\pi^3}{\sqrt{7}}\right)^{15}$

$y' = 0$

Example: Find the derivatives of these functions.

A) $y = 5, \quad y = \sqrt{8}, \quad y = \pi^4, \quad y = \sin(20^o)$

$y' = 0$

Compute $\boxed{\frac{dy}{dA}}$ for $y = 3x^2$

$\frac{dy}{dA} = 0$

Compute y' for $y = 3x^2$

$y' = 6x$

B) $y = x^{10}$ $\quad$ $y' = 10x^9$

C) $y = 3x^5$ $\quad$ $y' = 3 \cdot 5x^4 = 15x^4$

D) $B(x) = 3 - 7x^{1} + 4x^5$

$B'(x) = 0 - 7 \cdot 1x^0 + 4 \cdot 5x^4$

$B'(x) = -7 + 20x^4$

a is a #.

Examine the derivative of $f(x) = a^x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h\to 0} \frac{(a^h - 1)}{h}$$

For $a = 2$ then $\lim_{h\to 0} \frac{(2^h - 1)}{h} = 0.69$ and for $a = 3$ then $\lim_{h\to 0} \frac{(3^h - 1)}{h} = 1.10$.

Thus by the Intermediate value theorem, there is a number between 2 and 3 such that

$\lim_{h\to 0} \frac{(a^h - 1)}{h} = 1$. This number is $e = 2.71828.....$

$f(x) = e^x$

$f'(x) = e^x$

Example: Find the indicated derivative of these functions.

A) y' if $y = 3e^x + 2x^e$

$$\frac{dy}{dx} = y' = 3 \cdot e^x + 2 \cdot e\, x^{e-1}$$

B) y'' if $y = \sqrt[3]{x^8} + \sqrt{x^5} + e^{x+4}$ $= x^{8/3} + x^{5/2} + e^x e^4$

$$y' = \frac{8}{3} x^{\frac{8}{3}-1} + \frac{5}{2} x^{\frac{5}{2}-1} + e^x e^4$$

$$y' = \frac{8}{3} x^{5/3} + \frac{5}{2} x^{3/2} + e^x e^4$$

$$y'' = \frac{8}{3} \cdot \frac{5}{3} x^{2/3} + \frac{5}{2} \cdot \frac{3}{2} x^{1/2} + e^x e^4$$

$$y'' = \frac{40}{9} x^{2/3} + \frac{15}{4} x^{1/2} + e^x e^4$$

$$\frac{8}{3} - 1 = \frac{8}{3} - \frac{3}{3} = \frac{5}{3}$$

$$\frac{5}{2} - 1 = \frac{5}{2} - \frac{2}{2} = \frac{3}{2}$$

1st deriv	2nd deriv.
y'	y''
$f'(x)$	$f''(x)$
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$

C) $f'''(x)$ if $f(x) = 3x^6 + 2x + 5$

$$f'(x) = 18x^5 + 2$$

$$f''(x) = 90x^4$$

$$f'''(x) = 360x^3$$

4th deriv.

$y^{(4)}$ $\quad \frac{d^4y}{dx^4}$

$f^{(4)}(x)$

D) y' if $y = 3a^{-5} + \frac{1}{2a^3} + 3^8$ $= 3a^{-5} + \frac{1}{2}a^{-3} + \underbrace{3^8}$ Constant.

$y' = -15a^{-6} + \frac{1}{2}(-3)a^{-4} + 0$

$= -15a^{-6} - \frac{3}{2}a^{-4}$

E) y' if $y = \frac{m^3 + 5m^2 + 7}{m}$ $= \frac{m^3}{m} + \frac{5m^2}{m} + \frac{7}{m}$

$y = m^2 + 5m + 7m^{-1}$

$y' = 2m + 5 - 7m^{-2}$

F) y' if $y = \frac{x^4+1}{x^2\sqrt{x}}$ $= \frac{x^4+1}{x^{2.5}} = \frac{x^4}{x^{2.5}} + \frac{1}{x^{2.5}}$

$= x^{1.5} + x^{-2.5}$

$y' = 1.5x^{.5} - 2.5x^{-3.5} = \frac{3}{2}x^{1/2} - \frac{5}{2}x^{-7/2}$

Example: Find the equation of the tangent line and the normal line to $f(x) = x^2 + 5x + 10$ at $x = 3$

point.

$f(3) = 9 + 15 + 10 = 34$

m_{tan} = inst. Rate of Change. $= f'(3)$

$f'(x) = 2x + 5$

$m_{tan} = f'(3) = 2(3) + 5 = 11$

eq. of the tangent line

$y - f(3) = f'(3)(x-3)$

$y - 34 = 11(x-3)$

The normal line is the line perpendicular to the tangent line at the point.

$m_{normal} = \dfrac{-1}{m_{tan}} = \dfrac{-1}{11}$

$y - 34 = \dfrac{-1}{11}(x-3)$

Example: Find the value(s) of x where $f(x)$ has a tangent line that is parallel to $y = 6x + 5$

$m_{tan} = 6$

$f(x) = x^3 - 5x^2 + 6x - 30$

$f'(x) = 3x^2 - 10x + 6$

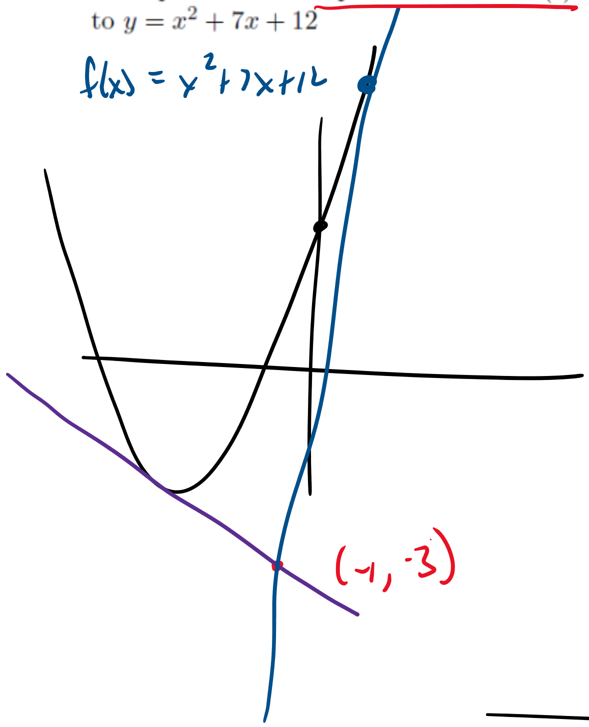
$3x^2 - 10x + 6 = 6$

$3x^2 - 10x = 0$

$x(3x - 10) = 0$

$\boxed{x = 0 \quad \Big| \quad x = \frac{10}{3}}$

Example: Find the equation of the line(s) thru the point $(-1, -3)$ that are tangent to $y = x^2 + 7x + 12$



This point is not on the graph.

equation of the tangent line at $x = A$

$y - y_1 = m(x - x_1)$

$y - f(A) = f'(A)(x - A)$

Point
$(A, f(A))$

Slope
$f'(A) = m$

$f(A) = A^2 + 7A + 12$

$f'(x) = 2x + 7$

$f'(A) = 2A + 7$

$y - (A^2 + 7A + 12) = (2A + 7)(x - A)$

$-3 - A^2 - 7A - 12 = (2A + 7)(-1 - A)$

$-A^2 - 7A - 15 = -2A - 2A^2 - 7 - 7A$

$A^2 + 2A - 8 = 0$

$(A + 4)(A - 2)$

$f'(x) = 2x + 7$

$\boxed{A = -4 \qquad A = 2}$

$f'(-4) = -8 + 7$
$= -1$

$f'(2) = 4 + 7 = 11$

$y + 3 = -1(x + 1)$

$y + 3 = 11(x + 1)$

Example: Find $g'(x)$ when $g(x) = \begin{cases} 1-2x & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 1 \\ x & \text{if } x > 1 \end{cases}$

Step 1

$$g'(x) = \begin{cases} -2, & x < -1 \\ 2x, & \boxed{-1 < x < 1} \\ 1, & x > 1 \end{cases}$$

Step 2

* Is g(x) continuous at the Break values.

x = -1

$$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} 1 - 2x = 1 + 2 = 3$$

$$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} x^2 = 1$$

not equal so g(x) is not cont at x = -1

So not diff. at x = -1

x = 1

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x^2 = 1$$

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} x = 1$$

equal so g(x) is cont. at x = 1

Step 3

* Is the function smooth at the Break value

↳ no sharp point.

need

$$\underbrace{\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} \text{ and } \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}}$$ to be equal.

↳ Translation is that the derivative is cont. at x = 1

$$\lim_{x \to 1^-} g'(x) = \lim_{x \to 1^-} 2x = 2$$

$$\lim g'(x) = \lim 1 = 1$$

not equal slopes do not match up

So g(x) has a sharp

$$\lim_{x \to 1^+} g'(x) = \lim_{x \to 1^+} 1 = 1$$

So $g(x)$ has a sharp point at $x=1$

So not diff. at $x=1$

Answer

$$g'(x) = \begin{cases} -2, & x < -1 \\ 2x, & -1 < x < 1 \\ 1, & x > 1 \end{cases}$$

Example: Find $k'(x)$ when $k(x) = \begin{cases} 4x^2 + 2x + 4 & \text{if } x < 1 \\ 10x - 3 & \text{if } x \geq 1 \end{cases}$

Answer

$$K'(x) = \begin{cases} 8x + 2, & x < 1 \\ 10, & x > 1 \end{cases}$$

$$\lim_{x \to 1^-} K(x) = \lim_{x \to 1^-} 4x^2 + 2x + 4$$
$$= 4 + 2 + 4 = 10$$

$$\lim_{x \to 1^+} K(x) = \lim_{x \to 1^+} 10x - 3 = 7$$

K(x) is not cont at x=1
So not diff.

Example: Find $k'(x)$ when $k(x) = \begin{cases} 4x^2 + 2x + 1 & \text{if } x < 1 \\ 10x - 3 & \text{if } x \geq 1 \end{cases}$

Answer

$$K'(x) = \begin{cases} 8x + 2, & x < 1 \\ 10, & x \geq 1 \end{cases}$$

$$\lim_{x \to 1^-} K(x) = 4 + 2 + 1 = 7$$
$$\lim_{x \to 1^+} K(x) = 10 - 3 = 7$$

cont.

$$\lim_{x \to 1^-} K'(x) = 8(1) + 2 = 10$$
$$\lim_{x \to 1^+} K'(x) = 10$$

Smooth graph.