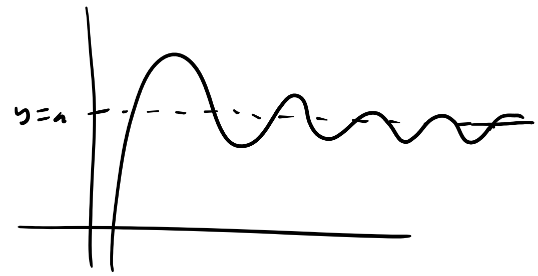
Section 2.6: Limits at Infinity

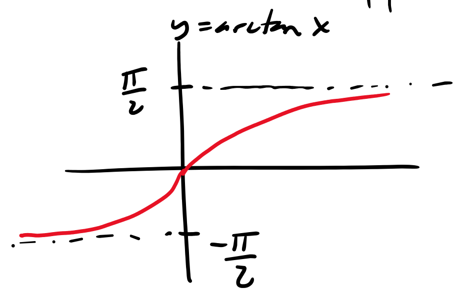
The end behavior of a function is computed by $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. If either of these limits is a number, L, then $y = L$ is called a **horizontal asymptote** of $f(x)$.



Example: Compute these limits.

A) $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$

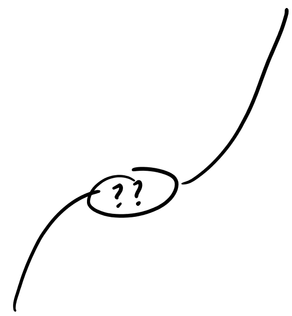
B) $\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$



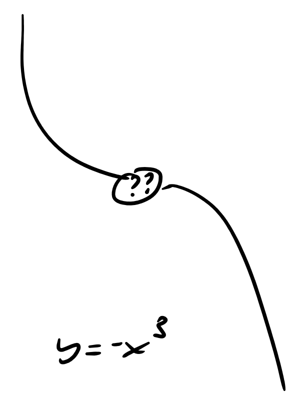
C) $\lim_{x\to\infty} (x^2 - 4x + 2) = +\infty$

∪

D) $\lim_{x\to\infty} (x^2 - x^5) = -\infty$



$y = x^3$



$y = -x^3$

E) $\lim_{x\to\infty}\left[3-2\left(\frac{\pi}{4}\right)^x\right] = 3-2(0) = 3$

$\frac{\pi}{4}<1$ decay.

F) $\lim_{x\to-\infty}\left[3-2\left(\frac{\pi}{4}\right)^x\right] = -\infty$

as $x\to-\infty$ $\left(\frac{\pi}{4}\right)^x \to +\infty$

$y=a^x$

$a>1$ $\quad$ $0<a<1$

growth $K>0$ $\quad$ decay $K<0$

e^{Kx}

G) $\lim_{x\to\infty}\frac{80}{4+2e^{-0.15x}} = \frac{80}{4+2(0)} = \frac{80}{4} = 20$

decay

H) $\lim_{x\to-\infty}\frac{80}{4+2e^{-0.15x}} = 0$

decay $\quad$ as $x\to-\infty$ $\quad$ $e^{-.15x}\to\infty$ $\quad$ $4+2e^{-.15x}\to\infty$

Theorem: If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \text{ and } \lim_{x\to-\infty} \frac{1}{x^r} = 0$$

Example: Compute these limits.

A) $\lim\limits_{x\to\infty} \dfrac{3x^2+1}{2x^3+x^2} = \lim\limits_{x\to\infty} \dfrac{x^3\left(\frac{3x^2}{x^3}+\frac{1}{x^3}\right)}{x^3\left(2+\frac{x^2}{x^3}\right)} = \lim\limits_{x\to\infty} \dfrac{x^3\left(\frac{3}{x}+\frac{1}{x^3}\right)}{x^3\left(2+\frac{1}{x}\right)}$

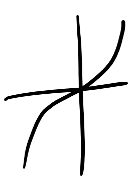
$= \lim\limits_{x\to\infty} \dfrac{\frac{3}{x}+\frac{1}{x^3}}{2+\frac{1}{x}} = \dfrac{0+0}{2+0} = 0$

The function $f(x) = \dfrac{3x^2+1}{2x^3+x^2}$ has a horizontal asymptote of $y=0$

B) $\lim\limits_{x\to-\infty} \dfrac{3x^4+7}{x+2} = \lim\limits_{x\to-\infty} \dfrac{x\left(3x^3+\frac{7}{x}\right)}{x\left(1+\frac{2}{x}\right)} = \lim\limits_{x\to-\infty} \dfrac{3x^3+\frac{2}{x}}{1+\frac{2}{x}} = -\infty$

$-\infty$ $\to 0$ $\to 0$

as $x \to -\infty$



$3x^3 \to -\infty$

C) $\lim_{x\to\infty} \frac{\sqrt{2x^2+3}}{x+2} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(2+\frac{3}{x^2}\right)}}{x\left(1+\frac{2}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{2+\frac{3}{x^2}}}{x\left(1+\frac{2}{x}\right)}$

$$\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$= \lim_{x\to\infty} \frac{x\sqrt{2+\frac{3}{x^2}}}{x\left(1+\frac{2}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{2+\frac{3}{x^2}}}{1+\frac{2}{x}} = \frac{\sqrt{2+0}}{1+0} = \sqrt{2}$$

D) $\lim_{x\to-\infty} \frac{\sqrt{2x^2+3}}{x+2} = \lim_{x\to-\infty} \frac{\sqrt{x^2\left(2+\frac{3}{x^2}\right)}}{x\left(1+\frac{2}{x}\right)} = \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{2+\frac{3}{x^2}}}{x\left(1+\frac{2}{x}\right)}$

$\sqrt{x^2} = |x| = -x$

for $x \to -\infty$

$= \lim_{x\to-\infty} \frac{-x\sqrt{2+\frac{3}{x^2}}}{x\left(1+\frac{2}{x}\right)}$

$= \lim_{x\to-\infty} \frac{-\sqrt{2+\frac{3}{x^2}}}{1+\frac{2}{x}} = \frac{-\sqrt{2+0}}{1+0} = -\sqrt{2}$

Case $\infty - \infty$

E) $\lim\limits_{x\to\infty} \left(\sqrt{x^2+3x} - x\right) \cdot \dfrac{\left(\sqrt{x^2+3x} + x\right)}{\sqrt{x^2+3x} + x} = \lim\limits_{x\to\infty} \dfrac{x^2+3x - x^2}{\sqrt{x^2+3x} + x}$

x^2+3x

$$= \lim_{x\to\infty} \frac{3x}{\sqrt{x^2+3x} + x} = \lim_{x\to\infty} \frac{3x}{\sqrt{x^2\left(1+\frac{3}{x}\right)} + x}$$

$$= \lim_{x\to\infty} \frac{3x}{x\sqrt{1+\frac{3}{x}} + x}$$

$$= \lim_{x\to\infty} \frac{3x}{x\left[\sqrt{1+\frac{3}{x}} + 1\right]} = \lim_{x\to\infty} \frac{3}{\sqrt{1+\frac{3}{x}} + 1}$$

$$= \frac{3}{\sqrt{1+0} + 1} = \frac{3}{\sqrt{1}+1} = \frac{3}{2}$$

F) $\lim_{x \to -\infty} \frac{\sqrt{3x^4+1}}{2x^2+7} = \lim_{x \to -\infty} \frac{\sqrt{x^4\left(3+\frac{1}{x^4}\right)}}{x^2\left(2+\frac{7}{x^2}\right)} = \lim_{x \to -\infty} \frac{\sqrt{x^4}\sqrt{3+\frac{1}{x^4}}}{x^2\left(2+\frac{7}{x^2}\right)}$

$\sqrt{x^4} = \sqrt{x^2} \cdot \sqrt{x^2}$

$= |x| \cdot |x| = (-x)(-x)$

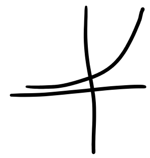
as $x \to -\infty$ $\quad = x^2$

$= \lim_{x \to -\infty} \frac{x^2\sqrt{3+\frac{1}{x^4}}}{x^2\left(2+\frac{7}{x^2}\right)}$

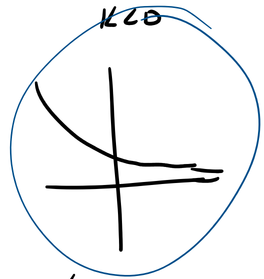
$= \lim_{x \to -\infty} \frac{\sqrt{3+\frac{1}{x^4}}}{2+\frac{7}{x^2}} = \frac{\sqrt{3+0}}{2+0} = \frac{\sqrt{3}}{2}$

e^{kx}

exp growth $k>0$



decay $k<0$



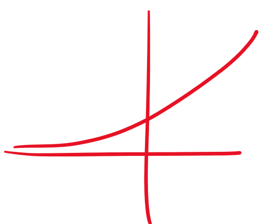
G) $\underbrace{\lim_{x\to\infty}}_{\text{want exp. decay.}} \dfrac{2e^{3x}+e^{-2x}}{3e^{4x}+5e^{-2x}} = \lim_{x\to\infty} \dfrac{e^{4x}\left(\frac{2e^{3x}}{e^{4x}}+\frac{e^{-2x}}{e^{4x}}\right)}{e^{4x}\left(3+\frac{5e^{-2x}}{e^{4x}}\right)}$

$$= \lim_{x\to\infty} \frac{e^{4x}\left(2e^{-x}+e^{-6x}\right)}{e^{4x}\left(3+5e^{-6x}\right)} = \lim_{x\to\infty} \frac{2e^{-x}+e^{-6x}}{3+5e^{-6x}} = \frac{2(0)+0}{3+5(0)}$$

$$= 0$$

$$\boxed{\frac{e^{3x}}{e^{-2x}} = e^{3x}e^{2x} = e^{5x}}$$

H) $\underbrace{\lim_{x\to-\infty}}_{\text{growth}} \dfrac{2e^{3x}+e^{-2x}}{3e^{4x}+5e^{-2x}} = \lim_{x\to-\infty} \dfrac{e^{-2x}\left(\frac{2e^{3x}}{e^{-2x}}+1\right)}{e^{-2x}\left(\frac{3e^{4x}}{e^{-2x}}+5\right)}$



e^{kx} $k>0$

$$= \lim_{x\to-\infty} \frac{\cancel{e^{-2x}}\left(2e^{5x}+1\right)}{\cancel{e^{-2x}}\left(3e^{6x}+5\right)} = \lim_{x\to-\infty} \frac{2e^{5x}+1}{3e^{6x}+5}$$

$$= \frac{0+1}{0+5} = \frac{1}{5}$$

case $\infty - \infty$

I) $\lim_{x\to\infty} \left[\ln(3x+5) - \ln(2+5x)\right]$

$$= \lim_{x\to\infty} \ln\left(\frac{3x+5}{2+5x}\right)$$

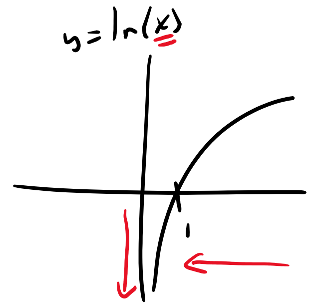
$$= \ln\left(\underbrace{\lim_{x\to\infty} \frac{3x+5}{2+5x}}\right) = \ln\left(\frac{3}{5}\right)$$

$$\lim_{x\to\infty} \frac{3x+5}{2+5x} = \lim_{x\to\infty} \frac{\cancel{x}\left(3+\frac{5}{x}\right)}{\cancel{x}\left(\frac{2}{x}+5\right)} = \lim_{x\to\infty} \frac{3+\frac{5}{x}}{\frac{2}{x}+5} = \frac{3+0}{0+5} = \frac{3}{5}$$

J) $\lim\limits_{x\to\infty}\left[\ln(x+5)-\ln(2+x^2)\right] = \lim\limits_{x\to\infty} \ln\left(\frac{x+5}{2+x^2}\right)$

$$= \ln\left(\underbrace{\lim_{x\to\infty}\left(\frac{x+5}{2+x^2}\right)}\right)$$

$$= -\infty$$



$$\lim_{x\to\infty}\frac{x+5}{2+x^2} = \lim_{x\to\infty}\frac{x^2\left(\frac{1}{x}+\frac{5}{x^2}\right)}{x^2\left(\frac{2}{x^2}+1\right)} = \lim_{x\to\infty}\frac{\frac{1}{x}+\frac{5}{x^2}}{\frac{2}{x^2}+1} = 0$$

Example: Find the horizontal asymptotes of these functions.

$$f(x) = \frac{x^4+3}{7x^5+8}$$

$$\lim_{x\to\infty} \frac{x^4+3}{7x^5+8} = \lim_{x\to\infty} \frac{\cancel{x^5}\left(\frac{1}{x}+\frac{3}{x^5}\right)}{\cancel{x^5}\left(7+\frac{8}{x^5}\right)} = 0$$

$$\lim_{x\to-\infty} \frac{\frac{1}{x}+\frac{3}{x^5}}{7+\frac{8}{x^5}} = \frac{0}{7} = 0$$

HA $y=0$

$$g(x) = \frac{x^4+3x^5}{7x^5+8}$$

$$\lim_{x\to\infty} \frac{\cancel{x^5}\left(\frac{1}{x}+3\right)}{\cancel{x^5}\left(7+\frac{8}{x^5}\right)} = \frac{3}{7}$$

HA $y=\frac{3}{7}$