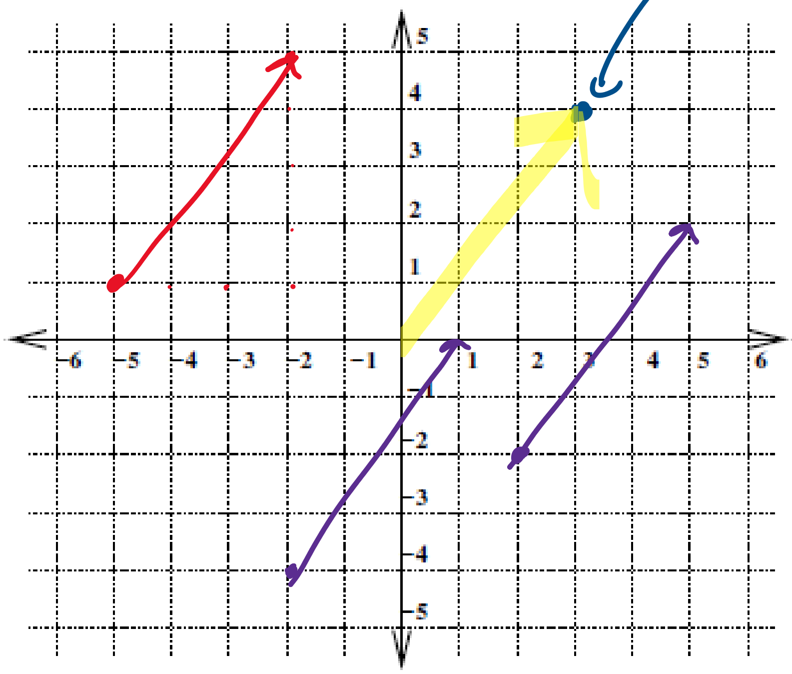
Appendix J.1: Vectors

Definition: A **vector** is a quantity that has both a magnitude and a direction. A **two-dimensional vector** is an ordered pair $\mathbf{a} = \langle a_1, a_2 \rangle$ of real numbers. The numbers a_1 and a_2 are called the **components** of **a**.

x y

Example: Graph the vector $\langle 3, 4 \rangle$.

point (3,4)

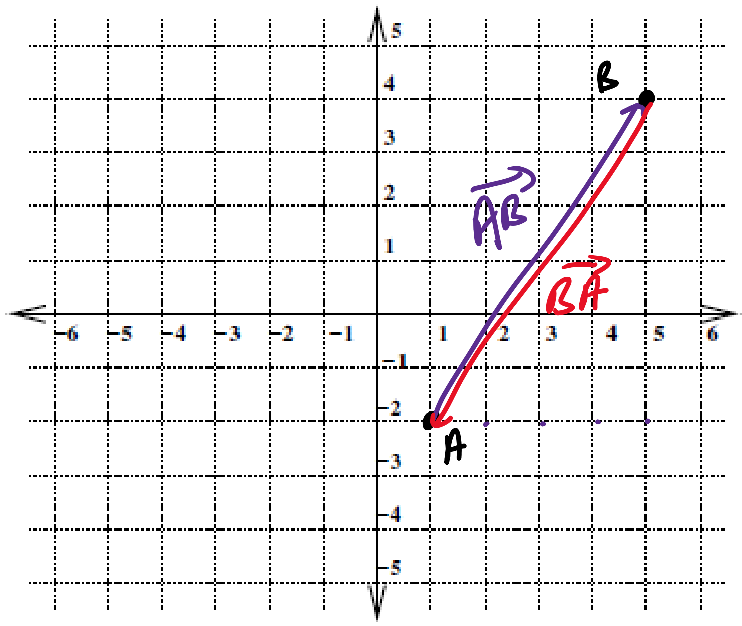


Example: For the points, $A(1,-2)$ and $B(5,4)$, find $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

$\overrightarrow{AB} = \langle 4, 6 \rangle$

start end.

$\overrightarrow{BA} = \langle -4, -6 \rangle$



Definition: Given two points $J(a_1, a_2)$ and $K(b_1, b_2)$, then the vector represended by

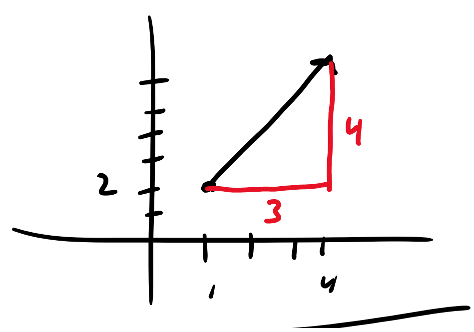
$\overrightarrow{JK} = \langle b_1 - a_1, b_2 - a_2 \rangle$

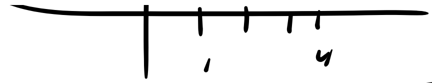
$\langle \text{end} - \text{start}, \text{end} - \text{start} \rangle$

Definition: The **magnitude** or **length** of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$|\mathbf{a}| = \sqrt{(a_1)^2 + (a_2)^2}$

$\langle 3, 4 \rangle$





$$|\langle 3,4\rangle| = \sqrt{3^2+4^2} = \sqrt{25}$$

$$= 5$$

Example: Find the length of these vectors.

A) $\langle 4, 6\rangle$ $\quad |\langle 4,6\rangle| = \sqrt{4^2+6^2} = \sqrt{16+36} = \sqrt{52}$

B) $\langle 0, 0\rangle$ $\quad |\langle 0,0\rangle| = \sqrt{0^2+0^2} = 0$

$5 = \sqrt{25} = \sqrt{(-5)^2} = -5$

Scalar Multiplication: If c is a scalar and $\mathbf{a} = \langle a_1, a_2\rangle$, then we define the vector $c\mathbf{a}$ as

$c\mathbf{a} = \langle ca_1, ca_2\rangle$ $\qquad 2\langle 3,4\rangle = \langle 6, 8\rangle$

$$|c\mathbf{a}| = \sqrt{(ca_1)^2+(ca_2)^2} = \sqrt{c^2a_1^2+c^2a_2^2}$$

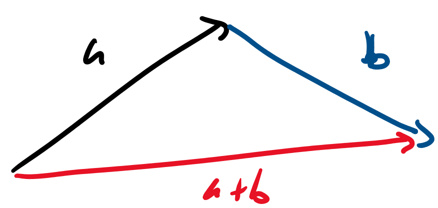
$$= \sqrt{c^2(a_1^2+a_2^2)} = \sqrt{c^2}\sqrt{a_1^2+a_2^2} = \underbrace{|c|}_{\text{absolute value}} \cdot \underbrace{|\vec{a}|}_{\text{magnitude or length of } \vec{a}}$$

Definition: Two vectors, $\mathbf{a}$ and $\mathbf{b}$ are said to be parallel if there is some scalar c such that $\mathbf{a} = c\mathbf{b}$

$\langle 3, 4 \rangle$ $\quad$ $\langle 9, 12 \rangle$

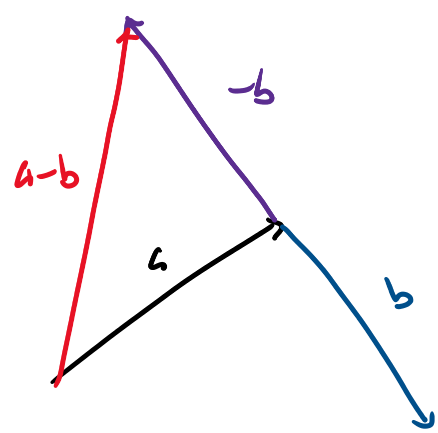
Vector Addition/Subtraction: If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ then the vector $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are defined as

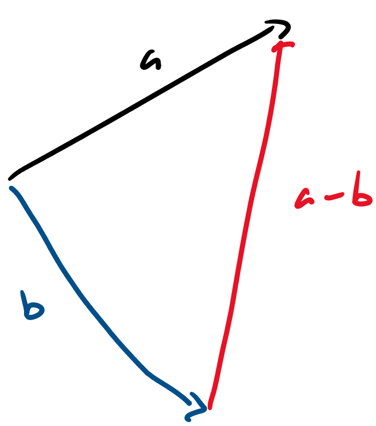
$\mathbf{a}+\mathbf{b} = \langle a_1 + b_1 , a_2 + b_2 \rangle$



$\mathbf{a}-\mathbf{b} = \langle a_1 - b_1 , a_2 - b_2 \rangle$

$a - b = a + (-1) b$





Example: Compute the following for $\mathbf{a} = \langle 3, 4\rangle$, $\mathbf{b} = \langle 6, 2\rangle$, $\mathbf{c} = \langle -2, 5\rangle$

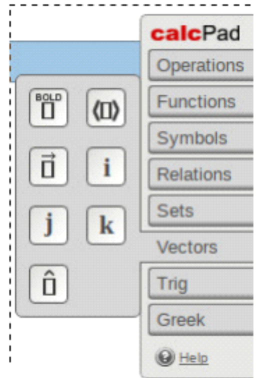
A) $\mathbf{a} + \mathbf{b}$ $= \langle 3,4\rangle + \langle 6,2\rangle = \langle 9,6\rangle$

B) $2\mathbf{a} + 3\mathbf{b}$ $= \langle 6,8\rangle + \langle 18,6\rangle = \langle 24, 14\rangle$

C) $\mathbf{a} - 2\mathbf{b}$ $= \langle 3,4\rangle - \langle 12,4\rangle = \langle -9, 0\rangle$

D) $3\mathbf{a} - 2\mathbf{c} + \mathbf{b}$ $= \langle 9, 12\rangle - \langle -4, 10\rangle + \langle 6.2\rangle = \langle 19, 4\rangle$

Definition: A unit vector is a vector of length 1. The vectors $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$ are referred to as the **standard basis vectors** for the xy plane.



$$\langle 3,4\rangle = 3i + 4j$$

Example: Find a vector of length 7 that is in the same direction as $\mathbf{a} = \langle 3, 4\rangle$

$$|a| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\frac{1}{5}a = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \leftarrow \text{unit vector.}$$

$\frac{1}{|a|}a$ I have normalized the vector a i.e. made it an unit vector

answer: $7 \cdot \frac{1}{5}a = \left\langle \frac{21}{5}, \frac{28}{5} \right\rangle$

$$\frac{7}{|a|}a =$$

end start

Example: Given the points $P(1,5)$ and $Q(3,10)$. Find a vector of length 4 that is in the same direction as $\overrightarrow{QP}$.

$$\overrightarrow{QP} = \langle 1-3, 5-10 \rangle = \langle -2, -5 \rangle$$

$$|\overrightarrow{QP}| = \sqrt{4+25} = \sqrt{29}$$

Answer is $\frac{4}{\sqrt{29}}\langle -2, -5 \rangle = \left\langle \frac{-8}{\sqrt{29}}, \frac{-20}{\sqrt{29}} \right\rangle$

W 30° N

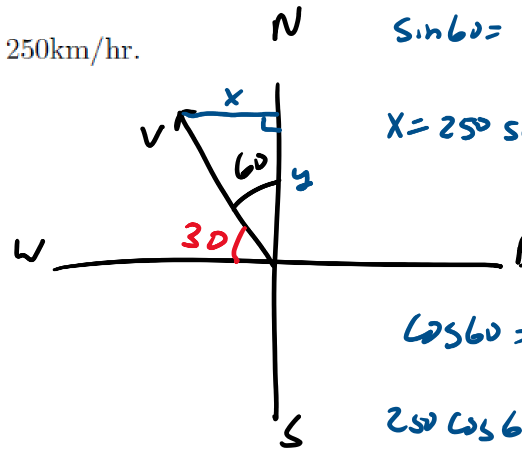
Example: A pilot is flying in the direction of N60^{o}W at a speed of 250km/hr.

A) Find the velocity vector.

$\sin 60 = \frac{x}{h}$

$\sin 60 = \frac{x}{250}$

$x = 250 \sin 60$



$\cos 60 = \frac{y}{h}$

$250 \cos 60 = y$

$V = \langle -250 \sin 60 \, , \, 250 \cos 60 \rangle$

$= \langle -250 \frac{\sqrt{3}}{2} \, , \, 250 \left(\frac{1}{2}\right) \rangle$

$V = \langle -125\sqrt{3} \, , \, 125 \rangle$

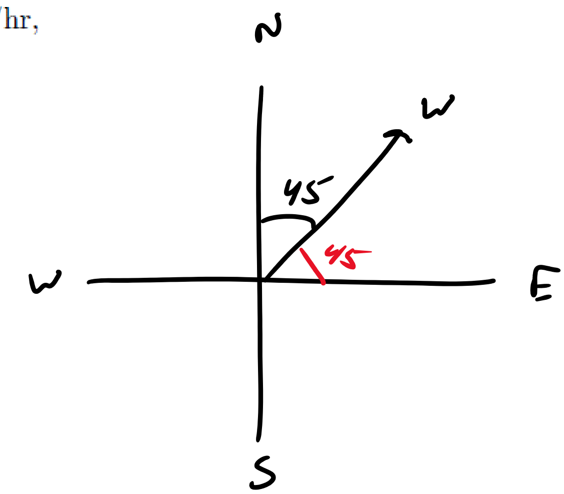
$V = \langle -250 \cos 30, \, 250 \sin 30 \rangle$

B) If there is a wind blowing in the direction of N45^{o}E at 30km/hr, find the true course and ground speed of the plane.

$W = \langle 30 \sin 45 \, , \, 30 \cos 45 \rangle$

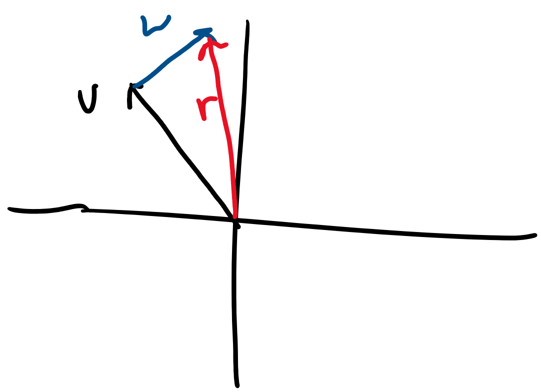
$= \langle 30 \frac{\sqrt{2}}{2} \, , \, 30 \frac{\sqrt{2}}{2} \rangle$

$= \langle 15\sqrt{2} \, , \, 15\sqrt{2} \rangle$



result:

$r = V + W$

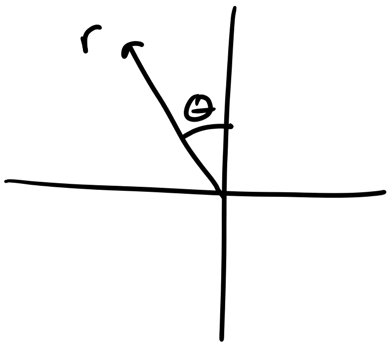


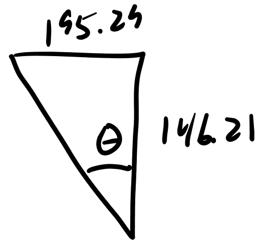
$= \langle -125\sqrt{3} \, , 125 \rangle + \langle 15\sqrt{2}, \, 15\sqrt{2} \rangle$

$= \langle -125\sqrt{3} + 15\sqrt{2} \, , \, 125 + 15\sqrt{2} \rangle$

$= \langle -195.29 \, , \, 146.21 \rangle$

$\text{Speed} = |r| = \sqrt{(-195.29)^2 + (146.21)^2} = 243.96 \text{ Km/hr.}$





$$\tan\theta = \frac{195.29}{146.21}$$

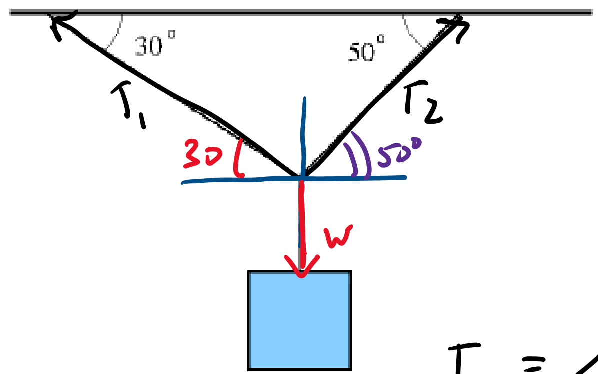
$$\theta = \arctan\left(\frac{195.29}{146.21}\right)$$

$$\theta = 53.18°$$

Bearing N 53.18° W

W 36.82 N

Example: A 50lb weight hangs from 2 wires. Find the tensions(forces) T_1 and T_2 in both wires and their magnitudes.



$$w = \langle 0, -50 \rangle$$

$$T_1 + T_2 + w = \vec{0}$$

$$T_1 = \langle -|T_1| \cos 30, \ |T_1| \sin 30 \rangle$$

$$T_2 = \langle |T_2| \cos 50, \ |T_2| \sin 50 \rangle$$

$$T_1 + T_2 + w = \vec{0}$$

$$-|T_1| \cos 30 + |T_2| \cos 50 + 0 = 0$$

$$|T_1| \sin 30 + |T_2| \sin 50 + -50 = 0$$

$$|T_1| \sin 30 + |T_2| \sin 50 = 50$$

$$|T_2| \cos 50 = |T_1| \cos 30$$

$$|T_2| = |T_1| \frac{\cos 30}{\cos 50}$$

$$|T_1| \sin 30 + |T_1| \frac{\cos 30}{\cos 50} \sin 50 = 50$$

$$|T_1| \left(\sin 30 + \frac{\cos 30}{\cos 50} \sin 50 \right) = 50$$

$$|T_1| = \frac{50}{\sin 30 + \frac{\cos 30}{\cos 50} \sin 50}$$

$$|T_1| = 32.64$$

$$|T_2| = 43.97$$

$$T_1 = \langle -28.27, \ 16.32 \rangle$$

$$T_2 = \langle 28.27, \ 33.68 \rangle$$