WIR_23A_M251_H8_spaced

Monday, April 3, 2023 5:46 PM

WIR_23A_M251_H8_spaced

MATHEMATICS
TEXAS A&M UNIVERSITY

Instructor: Rosanna Pearlstein

Math Learning Center

Math 251 – Spring 2023
"Week-in-Review"

Wir 8: Exam 3 Review

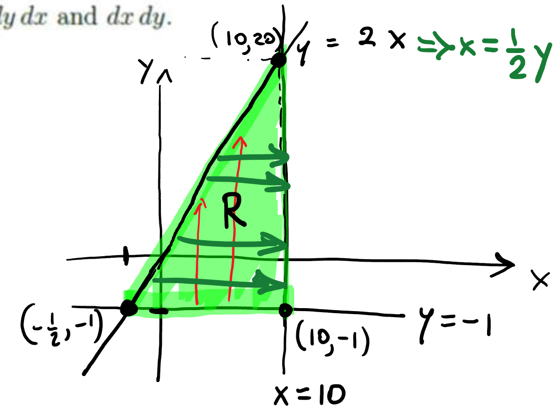
Sections 15.1-15.4, 15.6-15.9

Problem 1. Let R be the region in the xy–plane bounded by $y = 2x$, $x = 10$, and $y = -1$. Set up but do not evaluate $\displaystyle\iint_R (x^2 + y^2)\, dA$ in the order $dy\, dx$ and $dx\, dy$.

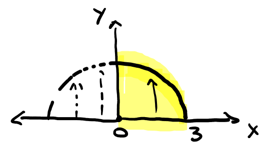
type I

$$\begin{cases} -\frac{1}{2} \le x \le 10 \\ -1 \le y \le 2x \end{cases}$$

$$\int_{-\frac{1}{2}}^{10}\int_{-1}^{2x} (x^2+y^2)\, dy\, dx = \int_{-1}^{20}\int_{\frac{1}{2}y}^{10} (x^2+y^2)\, dx\, dy$$



Problem 2. Evaluate $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} e^{-x^2-y^2}\, dy\, dx$



$$\boxed{\int_0^{\frac{\pi}{2}}} \int_0^3 e^{-r^2}\, r\, dr\, \boxed{d\theta}$$

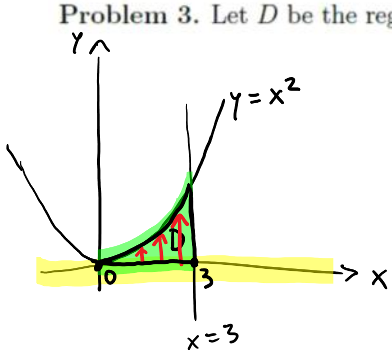
$u = -r^2$

$du = -2r\, dr$

$$-\int_{-0^2}^{-(3^2)} \frac{1}{2} e^u\, du = \frac{1}{2}\int_{-9}^{0} e^u\, du = \frac{1}{2}\left(e^0 - e^{-9}\right) = \frac{1}{2}\left(1 - \frac{1}{e^9}\right)$$

$$\frac{\pi}{4}\left(1 - \frac{1}{e^9}\right).$$

Problem 3. Let D be the region bounded by $y = 0$, $y = x^2$, and $x = 3$. Find $\displaystyle\iint_D 3x\cos y\,dA$.



$$\int_0^3 \int_0^{x^2} 3x\cos y \, dy\, dx =$$

Instructor: Rosanna Pearlstein

Math Learning Center

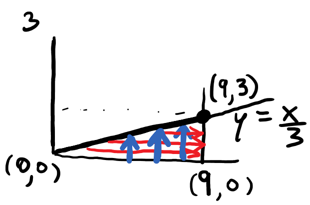
Math 251 – Spring 2023
"Week-in-Review"

$x = 3y \Leftrightarrow y = \frac{x}{3}$

Problem 4. Compute $\int_0^3 \int_{3y}^9 7e^{x^2}\, dx\, dy.$ $= \int_0^9 \left\{ \int_0^{\frac{x}{3}} e^{x^2}\, dy \right\} dx$

$\left. y e^{x^2} \right|_{y=0}^{\frac{x}{3}} =$

$\begin{cases} 3y \le x \le 9 \\ 0 \le y \le 3 \end{cases}$



$0 \le x \le 9$

$0 \le y \le \frac{x}{3}$

$= 7\int_0^9 \frac{x}{3} e^{x^2} - 0 \; dx$

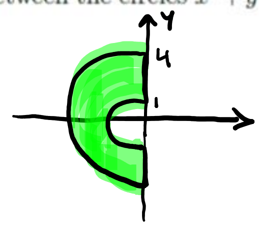
$7 \cdot \frac{1}{3} \cdot \frac{1}{2} \int_{0^2}^{9^2} e^u\, du =$

$u = x^2$

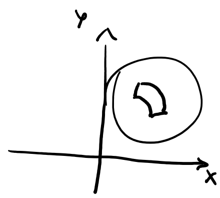
$du = 2x\, dx$

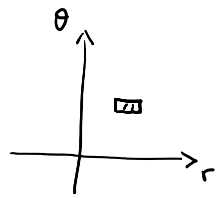
$7\left(\frac{1}{6} \left. e^u \right|_0^{81} \right) = \frac{7}{6}\left(e^{81} - 1 \right)$

Problem 5. Let R be the region that lies to the left of the y–axis between the circles $x^2+y^2=1$ and $x^2+y^2=16$. Find $\int\int_R 5(x+y)\,dA$



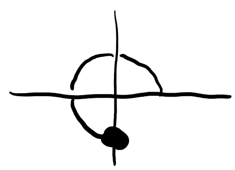
$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_1^4 5r(\cos\theta+\sin\theta)\,r\,dr\,d\theta =$$





$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(\cos\theta+\sin\theta)\,d\theta * \int_1^4 5r^2\,dr$$

$$\sin\theta-\cos\theta\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} * \frac{5r^3}{3}\Big|_1^4$$



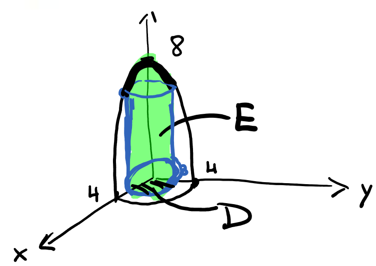
$$\left[-1-0-(1-0)\right]\left[\frac{5}{3}(4^3-1)\right]$$

$$= -2\,\frac{5}{3}(4^3-1) = -\frac{126}{3}*5 = \frac{-630}{3}$$

$$\underline{\underline{-210}}$$

$z = +\sqrt{64-4(x^2+y^2)}$

Problem 6. Find the volume of the solid that is above the xy plane, below the ellipsoid $4x^2+4y^2+z^2=64$ but inside the cylinder $x^2+y^2=9$.



$$\text{Vol } E = \iint_D \sqrt{64-4r^2}$$

$$\int_0^{2\pi}\int_0^3 \sqrt{64-4r^2}\; r\,dr\,d\theta$$

$$2\pi \int_0^3 2\sqrt{16-r^2}\; r\,dr \qquad 4(16-r^2)$$

$$u = 16 - r^2$$

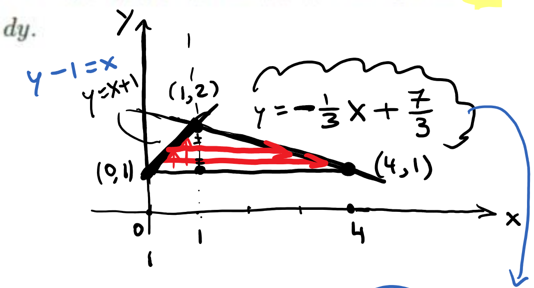
$$du = -2r\,dr$$

$$4\pi \int_{16-0}^{16-9} -\frac{1}{2} u^{1/2}\,du$$

$$2\cancel{4}\pi \left(-\frac{1}{\cancel{2}}\right)\frac{2}{3} u^{3/2}\Big|_{u=16}^{7} = -\frac{4}{3}\pi\left(7^{3/2}-4^3\right) =$$

$$= \frac{4}{3}\pi\left(4^3 - 7\sqrt{7}\right).$$

Problem 7. Let D be the triangular region with vertices $(0,1)$, $(1,2)$, and $(4,1)$. Set up but do not evaluate $\displaystyle\iint_D 7y^2\,dA$ in the order $dy\,dx$ and $dx\,dy$.



Type I

$$\int_0^1\int_1^{x+1} 7y^2\,dy\,dx + \int_1^4\int_1^{\frac{7}{3}-\frac{x}{3}} 7y^2\,dy\,dx$$

Type II

$$\int_1^2\int_{y-1}^{7-3y} 7y^2\,dx\,dy$$

$3y = -x + 7$

$x = 7 - 3y$

Problem 8. Let $D = \{(x, y) : 0 \le x \le 1, 0 \le y \le x^2\}$. Evaluate

$$\iint_D \frac{5y}{6x^5+1}\, dA.$$ dy dx (type I)

$$\int_0^1 \left\{ \int_0^{x^2} \frac{5y}{6x^5+1}\, dy \right\} dx \qquad \frac{5}{6x^5+1} \, \frac{1}{2}\left[y^2\right]_{y=0}^{x^2} = \frac{5}{2(6x^5+1)}\left(x^4 - 0\right)$$

$$\int_0^1 \frac{5}{2} \, \frac{x^4}{6x^5+1}\, dx \qquad u = 6x^5+1, \quad du = 30x^4\, dx$$

$6 \cdot 1 + 1 = 7$

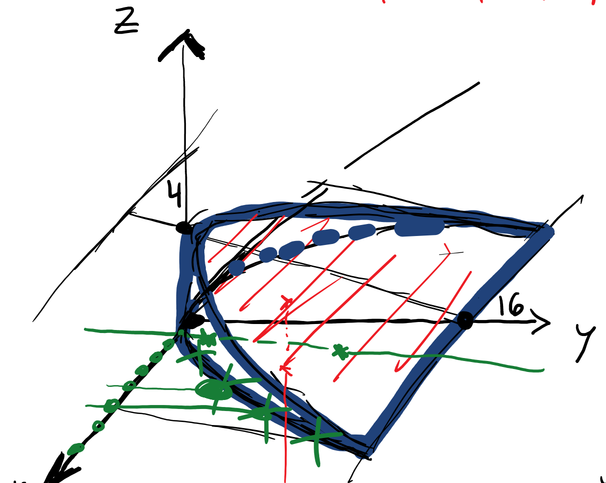
$$\int_1^7 \frac{\cancel{5}}{2} \, \frac{1}{\cancel{30}_{6}} \, \frac{1}{u}\, du = \frac{1}{12} \ln u \Big|_{u=1}^{7} = \boxed{\frac{1}{12} \ln 7} - \underbrace{\ln 1}_{=0}$$

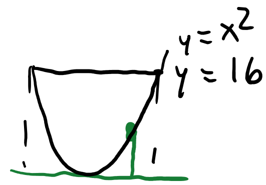
$6 \cdot 0 + 1 = 1$

Problem 9. Express $\iiint_E f(x,y,z)\,dV$ in the order $dydzdx$ if E is the solid bounded by $y = x^2$, $z = 0$, $y + 4z = 16$.

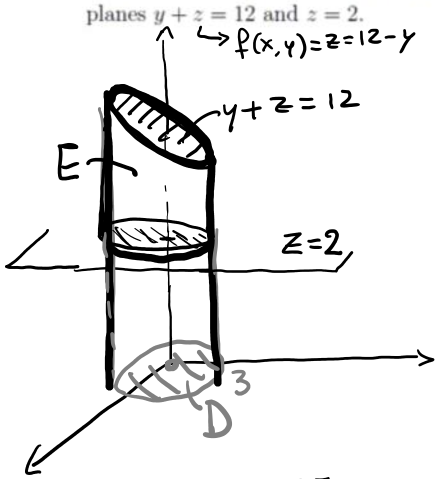
$\Rightarrow \frac{4z}{4} = \frac{16}{4} - \frac{y}{4}$

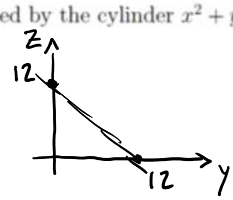
$$\int_{-4}^{4}\int_{0}^{4-\frac{1}{4}x^2}\int_{x^2}^{16-4z} f(x,y,z)\,dy\,dz\,dx$$





Problem 10. Find the volume of the solid that is enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 12$ and $z = 2$.





$$\iint_D [(12-y)-(2)]\,dA$$

$$\int_0^{2\pi}\int_0^{3} (10 - r\sin\theta)\, r\, dr\, d\theta \qquad \int 10r - r^2 \sin\theta \, dr$$

$$5r^2 - \frac{r^3}{3}\sin\theta \Big|_{r=0}^{3}$$

$$\int_0^{2\pi} 45 - 9\sin\theta - 0 \; d\theta = 45\theta + 9\cos\theta \Big|_0^{2\pi}$$

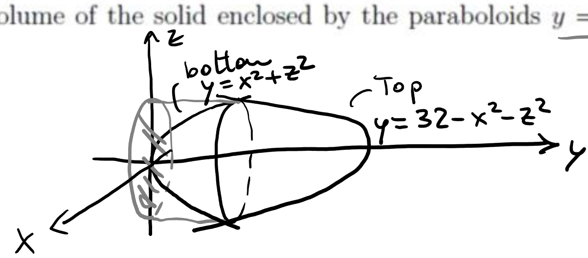
$$45\cdot(2\pi - 0) + 9(1-1)$$

$$90\pi.$$

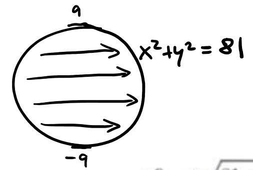
$r^2 = 32 - r^2$

$2r^2 = 32 \Rightarrow r^2 = 16 \Rightarrow r = 4$

Problem 11. Find the volume of the solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 32 - x^2 - z^2$.



$$\int_0^{2\pi}\int_0^4 \left[(32-r^2)-(r^2)\right] r\,dr\,d\theta$$



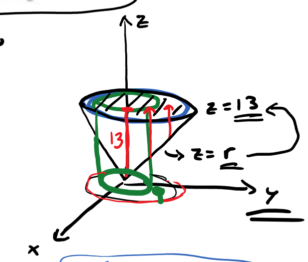
Problem 12. Convert to Cylindrical: $\displaystyle\int_{-9}^{9}\int_{-\sqrt{81-y^2}}^{\sqrt{81-y^2}}\int_{\sqrt{x^2+y^2}}^{13} \overset{r\cos\theta}{x}z\,dz\,dx\,dy.$

$z=\sqrt{x^2+y^2}$ cone

$\sqrt{x^2+y^2} \le z \le 13$

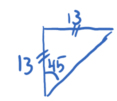
$$\begin{cases} r \le z \le 13 \\ 0 \le \theta \le 2\pi \\ 0 \le r \le 13 \end{cases}$$

$$\int_0^{2\pi}\int_0^{9}\int_r^{13} r\cos\theta\, z\, dz\, r\, dr\, d\theta$$



$$\boxed{\begin{array}{l}\text{cone: } z=r \\ \text{plane: } z=13\end{array}}$$

intersection $\Rightarrow r=13$



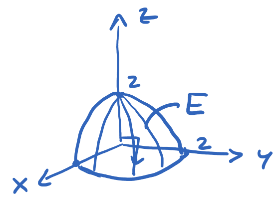
Problem 13. Find $\iiint_E \overbrace{(x^2+y^2+z^2)}^{\rho^2}\,dV$ where E is the part of the ball centered at the origin with radius 2 in the first octant.

sph. coord

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^2 \rho^2\, \rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta .$$

$$\frac{\pi}{2}\left[\frac{\rho^5}{5}\right]_0^2 \left[-\cos\varphi\right]_0^{\frac{\pi}{2}} =$$

$$= \frac{\pi}{2}\,\frac{2^5}{5}\left[-0 - -1\right] = \frac{16\pi}{5} .$$



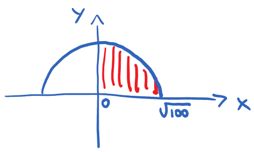
$$E : \begin{cases} 0 \le \rho \le 2 \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases} .$$

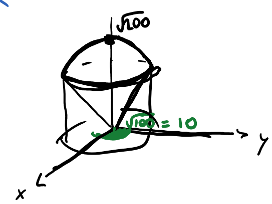
Instructor: Rosanna Pearlstein

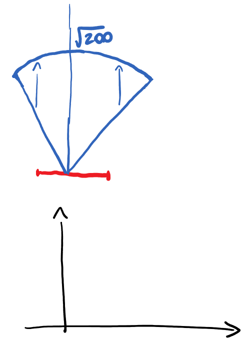
Math Learning Center

Math 251 – Spring 2023
“Week-in-Review”

Problem 14. Evaluate in spherical coordinates. $\displaystyle\int_0^{10}\int_0^{\sqrt{100-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{200-x^2-y^2}} yz\,dz\,dy\,dx$







$z = r$

$z = \sqrt{200 - r^2}$

$r^2 = 200 - r^2$

$2r^2 = 200$

$r^2 = 100$

$$\begin{cases} 0 \le \varphi \le \frac{\pi}{4} \\ 0 \le \rho \le \sqrt{200} \\ 0 \le \theta \le \frac{\pi}{2}. \end{cases}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{10\sqrt{2}} \rho\sin\varphi\sin\theta\,\rho\cos\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

$$\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta \int_0^{\frac{\pi}{4}}\sin^2\varphi\cos\varphi\,d\varphi \int_0^{10\sqrt{2}}\rho^4\,d\rho$$

$$-\cos\theta\Big|_0^{\frac{\pi}{2}} \quad \frac{1}{3}\sin^3\varphi\Big|_0^{\frac{\pi}{4}} \quad \frac{1}{5}\rho^5\Big|_0^{10\sqrt{2}}$$

$$\left(\frac{\sqrt{2}}{2}\right)^3 = \frac{\cancel{2}\sqrt{2}}{\cancel{8}_{\,4}}$$

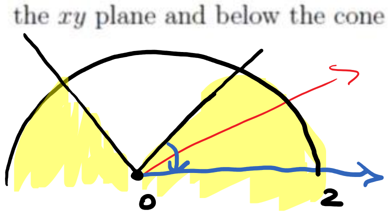
$$-(0-1)\left(\frac{1}{3}\right)\left(\frac{\sqrt{2}}{4} - 0\right)\frac{1}{5}\,10^5\,\sqrt{2}^5$$

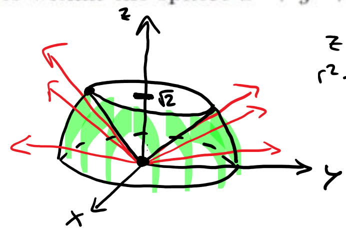
$$\frac{1}{3}\;\frac{\sqrt{2}}{4}\;\frac{1}{5}\;10^5\;\sqrt{2}^5\;..$$

Problem 15. Let E be the region that lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$. Set up but do not evaluate $\iiint_E (x+y+z)\,dV$ in spherical coordinates.

$$\int_0^{2\pi}\int_0^{\pi}\int_1^3 \rho\left(\sin\varphi\cos\theta+\sin\varphi\sin\theta+\cos\varphi\right)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

Problem 16. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy plane and below the cone $z = \sqrt{x^2 + y^2}$.





$$\left.\begin{array}{l} z = r \\ r^2 + z^2 = 4 \end{array}\right\}$$

$$r^2 + r^2 = 4$$
$$2r^2 = 4$$
$$r^2 = 2$$
$$r = \sqrt{2}$$

$\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}$

$0 \le \rho \le 2$

$0 \le \theta \le 2\pi$

$$Vol = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 1 \; \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$2\pi \; \frac{1}{3}\left[\rho^3\right]_0^2 \left[-\cos\varphi\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{2\pi}{3}(8-0)\left(-0 - -\frac{\sqrt{2}}{2}\right) =$$

$$= \frac{16\pi}{3}\frac{\sqrt{2}}{2}.$$

Instructor: Rosanna Pearlstein

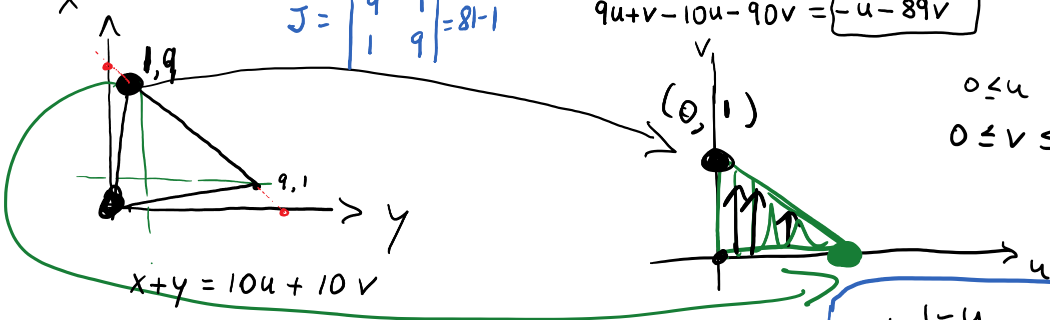
Math Learning Center

Math 251 – Spring 2023
"Week-in-Review"

Problem 17. Let R be the triangular region with vertices $(0,0)$, $(9,1)$, $(1,9)$. Using the transformation $x = 9u + v$ and $y = u + 9v$ find $\int\int_R (x - 10y)\,dA$.

$J = \begin{vmatrix} 9 & 1 \\ 1 & 9 \end{vmatrix} = 81 - 1$

$9u + v - 10u - 90v = \boxed{-u - 89v}$



$x + y = 10u + 10v$

$0 \le u \le 1$

$0 \le v \le 1 - u$

$$\boxed{\int_0^1 \int_0^{1-u} (-u - 89v)\,\Big|\underset{=80}{J}\Big|\,dv\,du.}$$

$x = 9u + v$

$(-9)\quad y = u + 9v$

$-9x = -81u - 9v$

$-9x + y = -80u + 0v$

$-9y = -9u - 81v$

$x - 9y = 0u - 80v$

$$\begin{cases} v = -\frac{1}{80}(x - 9y) \\ u = -\frac{1}{80}(-9x + y) \end{cases}$$

1, 9

9, 1

$v = -\frac{1}{80}(9 - 9) = 0$

$u = -\frac{1}{80}(-81 + 1) = 1$

$v = -\frac{1}{80}(1 - 81) = 1$

$u = -\frac{1}{80}(-9 + 9) = 0$

Problem 18. Let R be the parallelogram enclosed by the lines $x-6y=0$, $x-6y=9$, $6x-y=7$, $6x-y=10$. Using the transformation $u=x-6y$ and $v=6x-y$, find $\int\int_R 9\frac{x-6y}{6x-y}dA$

$$\sim J = \begin{vmatrix} 1 & -6 \\ 6 & -1 \end{vmatrix} = -1+36=35$$

$$\int_0^9\int_7^{10} 9^{\frac{u}{v}} \underbrace{|J|}_{=\frac{1}{35}} \, dv\, du$$

Problem 19. Let R be the region bounded by $25x^2 + 4y^2 = 100$. Using the transformation $x = 2u$ and $y = 5v$, find $\int\int_R 4x^2 dA$.

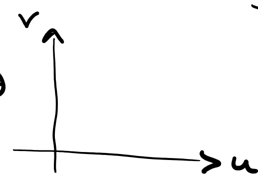
$25 \cdot 4u^2 + 4 \cdot 25v^2 = 100$

$u^2 + v^2 = 1$

$J = \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} = 10$

$$\iint_{\text{unit disk}} 4(4u^2)\,|\overset{=10}{J}|\,dA$$

$u = r\cos\theta$



polar

$$\int_0^{2\pi}\int_0^1 16(r\cos\theta)^2(10)\,r\,dr\,d\theta.$$

$$\overset{80}{\cancel{160}} \int_0^{2\pi} \overset{}{\cancel{\tfrac{1}{2}}}{}_1 (1+\cos 2\theta)\,d\theta \quad \int_0^1 r^3\,dr$$

$$\overset{20}{\cancel{80}}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} \quad \frac{1}{4},$$

$$20 \cdot 2\pi = 40\pi$$