MATHEMATICS
TEXAS A&M UNIVERSITY

Math Learning Center

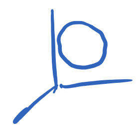
Instructor: Rosanna Pearlstein

Math 251 – Spring 2023
“Week-in-Review”

Chapters 12 and 13

Problem 1. What is the equation of the sphere centered at $(6, 4, 12)$ with radius 6? Describe the intersection of this sphere with the three coordinate planes.

$$(x-6)^2+(y-4)^2+(z-12)^2=6^2$$



$$(x-6)^2+(y-4)^2=-144+36$$

xy-plane: $(x-6)^2+(y-4)^2+(0-12)^2=36$ $z=0$ empty set

yz-plane: $(0-\cancel{6})^2+(y-4)^2+(z-12)^2=\cancel{3}6$ $x=0$ pt $(0,4,12)$.

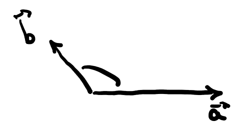
zx-plane: $(x-6)^2+(0-4)^2+(z-12)^2=36$ $y=0$

circle with radius $\sqrt{36-16}$ in the xz-plane.

$(6,0,12)$.

Problem 2. Let $\mathbf{a} = \langle 1, 2, -1\rangle$ and $\mathbf{b} = \langle 2, -1, 2\rangle$. Find the vector projection of $\mathbf{b}$ onto $\mathbf{a}$, that is $\text{proj}_{\mathbf{a}}\mathbf{b}$.

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{2-2-2}{\sqrt{1+4+1}} = \frac{-2}{\sqrt{6}}.$$



$$\text{proj}_{\vec{a}}\vec{b} = \frac{-2}{\sqrt{6}}\,\frac{\vec{a}}{|\vec{a}|} = \frac{-2}{\sqrt{6}}\,\frac{\vec{a}}{\sqrt{6}} = \frac{-2}{6}\langle 1, 2, -1\rangle = \langle -\tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{1}{3}\rangle.$$

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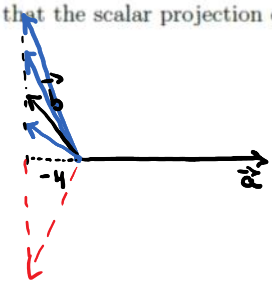
Problem 3. Let $\mathbf{a} = \langle -2, 2, 1\rangle$. Find a vector $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ so that the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ equals -4, that is $\text{comp}_{\mathbf{a}}\mathbf{b} = -4$.



$$\text{comp}_{\vec{a}}\vec{b} = -4 = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \Rightarrow$$

$$\Rightarrow \frac{-2b_1 + 2b_2 + b_3}{\sqrt{4+4+1}} = -4 \Rightarrow -2b_1 + 2b_2 + b_3 = -12$$

If $b_2 = b_3 = 0 \Rightarrow b_1 = 6$ $\quad \langle 6, 0, 0\rangle$

If $b_2 = 5$, $b_1 = 0 \Rightarrow b_3 = -12 - 10 = -22$

$\langle 0, 5, -22\rangle$.

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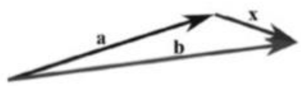
Problem 4. Use the figure below to answer the questions that follow.



a.) Write **x** in terms of **a** and **b**. $\vec{a} + \vec{x} = \vec{b} \Rightarrow \vec{x} = \vec{b} - \vec{a}$

b.) If the angle between **a** and **b** is 60°, $|\mathbf{a}| = 7$, and $|\mathbf{b}| = 6$, find $\mathbf{a} \cdot \mathbf{b}$. $= 7 \cdot 6 \cdot \cos \frac{\pi}{3} = 7 \cdot 6 \, \frac{1}{2} = 21$

c.) If the angle between **a** and **b** is 60°, $|\mathbf{a}| = 7$, and $|\mathbf{b}| = 6$, find $|\mathbf{a} \times \mathbf{b}|$ and determine whether $\mathbf{a} \times \mathbf{b}$ is directed into or out of the page.

$|\vec{a} \times \vec{b}| = 7 \cdot 6 \sin \frac{\pi}{3} = 7 \cdot 6 \, \frac{\sqrt{3}}{2} = 21\sqrt{3}$

into the page.

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Problem 5. Find a vector equation, a set of parametric equations, and symmetric equations for the line passing through the point $\underbrace{(-2, 3, 4)}_{P_0}$ that is parallel to the vector $\langle 1, -4, 4\rangle = \vec{v}$

$\langle x, y, z\rangle = \langle -2, 3, 4\rangle + \langle t, -4t, 4t\rangle$ vector equ.

parametric equs.

$$\begin{cases} x = -2 + t \\ y = 3 - 4t \\ z = 4 + 4t \end{cases} \Rightarrow t = \boxed{\frac{x+2}{1} = \frac{y-3}{-4} = \frac{z-4}{4}}$$ symmetric equs.

Problem 6. Consider the line that passes through the points $\overset{A}{(4,3,-1)}$ and $\overset{B}{(5,3,5)}$. Where does this line intersect the three coordinate planes, and if it does not intersect one of the three coordinate planes, explain why not.

$P_0\ (4,3,-1)$

$\vec{v} = \vec{AB} = B - A = \langle 1, 0, 6 \rangle$

$$\begin{cases} x = 4 + t \\ \boxed{y = 3} + 0t \\ z = -1 + 6t \end{cases}$$

xy-plane $z = 0 \Rightarrow t = \frac{1}{6} \Rightarrow x = 4 + \frac{1}{6}$

$(\frac{25}{6}, 3, 0)$

yz-plane $x = 0$ $t = -4$ $(0, 3, -25)$

xz-plane $y = 0$ never! NO intersection

P_0

Problem 7. Find the equation of the plane that contains the point $(1, 2, -5)$ and is perpendicular to the vector $\langle -6, 4, -2\rangle = \vec{n}$

$$-6(x-1)+4(y-2)-2(z+5)=0$$

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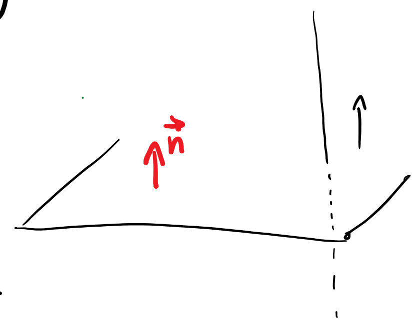
Problem 8. Find parametric equations for the line that passes through $\overset{P_0}{(2,-1,5)}$ and is

a.) parallel to the line $\dfrac{x+1}{3} = \dfrac{y-6}{4} = \dfrac{z}{1}$. $\vec{v} = \langle 3, 4, 1 \rangle$

b.) perpendicular to the plane $8x - 11y = 2z + 6$.

(a)
$$\begin{cases} x = 2 + 3t \\ y = -1 + 4t \\ z = 5 + t \end{cases}$$

(b) $\vec{n} = \langle 8, -11, -2 \rangle$



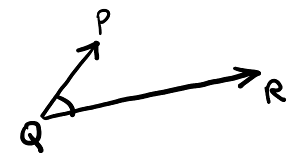
$$\begin{cases} x = 2 + 8t \\ y = -1 - 11t \\ z = 5 - 2t \end{cases}$$

Problem 9. Consider the triangle with vertices $\overset{P_0}{P(1,0,1)}$, $Q(2,3,4)$ and $R(2,1,1)$.

a.) Find the angle at the vertex Q.

b.) Find the equation of the plane that passes through the points

(a) $\cos\theta = \dfrac{\vec{QP}\cdot\vec{QR}}{|\vec{QP}|\,|\vec{QR}|} =$



$= \dfrac{0+6+9}{\sqrt{1+9+9}\,\sqrt{0+4+9}} = \dfrac{15}{\sqrt{19}\sqrt{13}}$

$\vec{QP} = \langle -1, -3, -3\rangle$

$\vec{QR} = \langle 0, -2, -3\rangle$

$\theta = \arccos \dfrac{15}{\sqrt{19}\sqrt{13}}$.

(b) $\vec{n} = \vec{PQ} \times \vec{RQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 3 \\ 0 & 2 & 3 \end{vmatrix} = \langle 3, -3, 2\rangle = \vec{n}$

$3(x-1) - 3(y-0) + 2(z-1) = 0$.

P_0

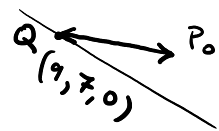
Problem 10. Find the equation of the plane that passes through the point (1, 0, 1) and

a.) is perpendicular to the line $x = 9 - t$, $y = 7 + 2t$, $z = t$.

b.) contains line $x = 9 - t$, $y = 7 + 2t$, $z = t$.

a) $\vec{n} = \langle -1, 2, 1 \rangle$

$$-1(x-1)+2(y-0)+1(z-1)=0$$



b) $\vec{n} = \langle -1, 2, 1 \rangle \times \vec{P_0Q}$

$\vec{P_0Q} \longrightarrow \langle 8, 7, -1 \rangle$

$$\vec{n} = \begin{vmatrix} i & j & k \\ -1 & 2 & 1 \\ 8 & 7 & -1 \end{vmatrix} = \langle -2-7, -(1-8), -7-16 \rangle =$$

$$= \langle -9, 7, -23 \rangle$$

$$-9(x-1)+7(y-0)-23(z-1)=0$$

Problem 11. Consider the plane P_1 given by the equation $2x - y + 3z = 7$ and the plane P_2 given by the equation $3x + y + 2z = 3$.

a.) Find the angle between the planes. $\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} = \frac{6 - 1 + 6}{\sqrt{4+1+9}\,\sqrt{9+1+4}} = \frac{11}{14} \Rightarrow \theta = \cos^{-1}\left(\frac{11}{14}\right).$

b.) Find a point, (x_0, y_0, z_0), that lies on both planes.

c.) Find a parametric equation for the line where the two planes intersect.

b) $\begin{cases} 2x - y + 3z = 7 \\ 3x + y + 2z = 3 \end{cases}$

$5x + 5z = 10$

If $x = 0 \Rightarrow z = 2$

$y = 3 - 0 - 4 = -1$

$P_0(0, -1, 2)$ on both planes

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

(c)

$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 3 & 1 & 2 \end{vmatrix} =$

$= \langle -5, -(4-9), 5 \rangle$

$= \langle 5, 5, 5 \rangle = \vec{v}$

$\begin{cases} x = 0 - 5t \\ y = -1 + 5t \\ z = 2 + 5t \end{cases}$

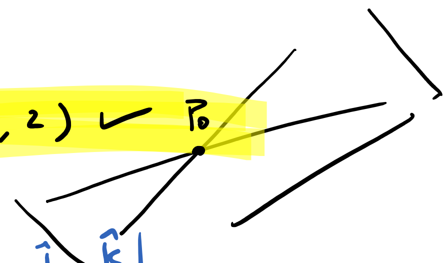
Problem 12. Consider the lines $\mathbf{r_1}(t) = \langle 1, 2, 0\rangle + t\,\langle 2, -2, 2\rangle$ and $\mathbf{r_2}(v) = \langle 3, 0, 2\rangle + v\,\langle -2, 2, 0\rangle$.

a.) Find the point where the two lines intersect.

b.) Find an equation of the plane containing both of these lines.

a)
$$\begin{cases} 1+2t = 3-2v \Rightarrow 1+2 = 3 = 3-2v \Rightarrow v = 0 \\ 2-2t = 0+2v \Rightarrow 0 = 2v \Rightarrow v = 0 \\ 2t = 2 \Rightarrow t = 1 \end{cases}$$

$(3, 0, 2)$ $(3, 0, 2)$ P_0



b)

$$\vec{n} = v_1 \times v_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 2 \\ -2 & 2 & 0 \end{vmatrix} = 2\cdot 2 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{vmatrix} =$$

$$= 4\langle -1, -(1), 0\rangle = 4\langle -1, -1, 0\rangle = \vec{n}$$

$$-1(x-3) - 1(y-0) + 0(z-2) = 0$$

$$-x - y + 3 = 0$$

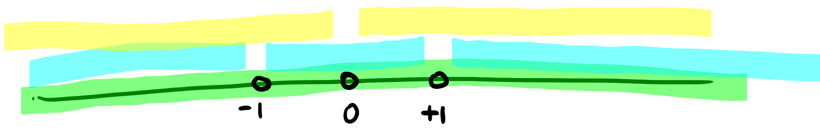
$1 \quad \frac{1}{2} \quad \frac{\sin 1}{1}$

$t^2 \neq 1 \Rightarrow t \neq \pm 1$

Problem 13. Let $\mathbf{r}(t) = \left\langle t^2, \frac{t-1}{t^2-1}, \frac{\sin t}{t} \right\rangle$.

a.) Find the domain of $\mathbf{r}(t)$.

b.) Find $\lim_{t\to 1} \mathbf{r}(t)$. $\left\langle 1, \frac{1}{2}, \sin 1 \right\rangle$



$\text{dom} = (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$

$\frac{\cancel{t-1}}{(\cancel{t-1})(t+1)} \longrightarrow \frac{1}{2}$

Problem 14. Let $\mathbf{r}(t) = \langle \cos(t^2), \sin(t^2), t^2 \rangle$.

a.) Find the velocity and speed of the curve at time $t = \sqrt{\pi}$.

b.) Find $\mathbf{T}(\sqrt{\pi})$, the unit tangent vector, at $t = \sqrt{\pi}$.

c.) Find $\mathbf{a}(t)$, the acceleration vector, at time t.

d.) The length of the curve from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

e.) The curvature of the curve traced out by $\mathbf{r}(t)$ when $t = \sqrt{\pi}$.

$t=0$ $\quad t=\sqrt{2\pi}$

$4t^2(\sin^2+\cos^2)$

a) $\vec{v}(t) = \vec{r}\,'(t) = \langle -2t\sin t^2, 2t\cos t^2, 2t \rangle$

$\vec{v}(\sqrt{\pi}) = \langle 0, 2\sqrt{\pi}(-1), 2\sqrt{\pi} \rangle = \langle 0, -2\sqrt{\pi}, 2\sqrt{\pi} \rangle. \rightarrow \dfrac{}{\sqrt{8\pi}}$

speed $= |\vec{v}(\sqrt{\pi})| = \sqrt{4\pi + 4\pi} = \sqrt{8\pi}$.

b) $\vec{T}(\sqrt{\pi}) = \dfrac{\vec{r}\,'(\sqrt{\pi})}{|\vec{r}\,'(\sqrt{\pi})|} = \left\langle 0, \dfrac{-2\sqrt{\pi}}{2\sqrt{2}\sqrt{\pi}}, \dfrac{2\sqrt{\pi}}{2\sqrt{2}\sqrt{\pi}} \right\rangle = \left\langle 0, \dfrac{-1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle$

c) $\vec{a}(t) = \vec{r}\,''(t) =$

$t = \sqrt{\pi}$

$= 2\langle -1\sin t^2 + (-t)2t\cos t^2, 1\cos t^2 + t[-2t\sin t^2], 1 \rangle = 2\langle +2\pi, -1, 1 \rangle$

d) $\mathcal{L} = \displaystyle\int_0^{\sqrt{2\pi}} \sqrt{4t^2 + 4t^2}\, dt = \int_0^{\sqrt{2\pi}} t\sqrt{8}\, dt = \sqrt{8}\,\frac{1}{2}t^2 \Big|_0^{\sqrt{2\pi}} =$

$= \dfrac{\sqrt{8}}{2}(2\pi - 0) = \pi\sqrt{8}$.

$k = \dfrac{|r' \times r''|}{|r'|^3} = \dfrac{4\sqrt{\pi}\,\sqrt{8\pi^2}}{\sqrt{8\pi}^{\,3}}$

$$2\begin{vmatrix} i & j & k \\ 0 & -2\sqrt{\pi} & 2\sqrt{\pi} \\ 2\pi & -1 & 1 \end{vmatrix} = 2\cdot 2\sqrt{\pi} \begin{vmatrix} i & j & k \\ 0 & -1 & 1 \\ 2\pi & -1 & 1 \end{vmatrix} =$$

$= 4\sqrt{\pi}\langle 0, --2\pi, 2\pi \rangle$

$|\vec{r}\,' \times \vec{r}\,''| = 4\sqrt{\pi}\sqrt{4\pi^2 + 4\pi^2} =$

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Problem 15. Find parametric equations for the tangent line to the curve $x = 4\sqrt{t}$, $y = t^2 - 10$, $z = \frac{4}{t}$ at $(8, 6, 1)$ P_0

$t=4$

$\vec{r}\,'(t) = \left\langle \frac{4}{2\sqrt{t}}, 2t, -\frac{4}{t^2} \right\rangle$

$\vec{r}\,'(t) = \left\langle 4\frac{1}{2\sqrt{t}}, 2t, -\frac{4}{t^2} \right\rangle =$

$\vec{r}\,'(4) = \langle 1, 8, -\frac{1}{4} \rangle$ $t=4$

$= \langle 1, 8, -\frac{1}{4} \rangle = \vec{v}$

$$\begin{cases} x = 8 + t \\ y = 6 + 8t \\ z = 1 - \frac{1}{4}t \end{cases}$$

Problem 16. If $\mathbf{r}'(t) = \langle t, e^t, te^{3t}\rangle$ and $\underbrace{\mathbf{r}(0) = \langle 1,3,2\rangle}$, find $\mathbf{r}(t)$. IVP

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt = \left\langle \frac{1}{2}t^2 + c_1,\ e^t + c_2,\ \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} + c_3 \right\rangle$$

der.	int
t (+)	e^{3t}
1 (−)	$\frac{1}{3}e^{3t}$
0	$\frac{1}{9}e^{3t}$

$t=0$

$$\vec{r}(0) = \langle 1,3,2\rangle = \left\langle c_1,\ 1+c_2,\ 0 - \frac{1}{9} + c_3 \right\rangle$$

$c_1 = 1$

$c_2 = 2$

$c_3 = \frac{19}{9}$

$2 + \frac{1}{9}$

$$\vec{r}(t) = \left\langle \frac{1}{2}t^2 + 1,\ e^t + 2,\ \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} + \frac{19}{9} \right\rangle.$$

Problem 17. Find $\int_0^1 \left(\frac{4t}{t^2+1}\mathbf{j} - \frac{1}{1+t^2}\mathbf{k}\right) dt.$ $= \langle 2\ln 2, -\frac{\pi}{4} \rangle$

$$\int \frac{4t}{t^2+1}\, dt \qquad u = t^2+1 \quad du = 2t\,dt$$

$$\int_{0+1}^{1+1} \frac{2\,du}{u} = 2\ln u \Big|_1^2 = 2(\ln 2 - \ln 1) = 2\ln 2$$

$$-\tan^{-1} t \Big|_0^1 = -\frac{\pi}{4} - 0$$

$t=0$ $\quad$ $v=0$

Problem 18. Given the curves $\mathbf{r}_1(t) = \langle 3t, t^2, t^3 \rangle$ and $\mathbf{r}_2(v) = \langle \sin v, \sin(2v), 6v \rangle$ intersect at the origin, find the angle of intersection.

$$r_1' = \langle 3, 2t, 3t^2 \rangle \to \langle 3, 0, 0 \rangle$$

$$r_2' = \langle \cos v, 2\cos 2v, 6 \rangle \longrightarrow \langle 1, 2, 6 \rangle$$

$$\cos\theta = \frac{\vec{r}_1{}' \cdot \vec{r}_2{}'}{|\vec{r}_1{}'|\,|\vec{r}_2{}'|} = \frac{3}{3\sqrt{1+4+36}} = \frac{1}{\sqrt{41}}$$

$$\theta = \arccos\frac{1}{\sqrt{41}}.$$

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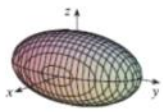
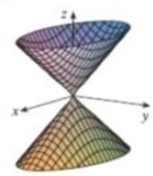
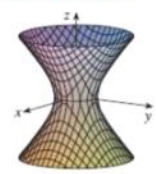
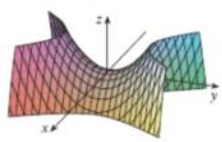
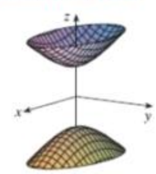
Problem 19. Be able to match an equation with the corresponding quadric surface.

Graphs of quadric surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Identify the following quadric surfaces:

$4x^2+9y^2-36z^2=36$ Hyperboloid of 1 sheet.

$16x^2+4y^2+4z^2-64x+8y+16z=0$

$$16x^2-64x \quad +4y^2+8y \quad +4z^2+16z \quad =0$$

$$\underbrace{16(x^2-4x+4)}+4(y^2+2y+1)+4(z^2+4z+4)=64+4+16$$

$$16(x-2)^2+4(y+1)^2+4(z+2)^2=84$$

$$16x^2+4y^2+4z^2=81 \quad \text{Ellipsoid.}$$

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$$-4x^2 + y^2 + 16z^2 - 8x + 10y + 32z = 0$$

$$-4x^2 - 8x \quad + y^2 + 10y \quad 16z^2 + 32z \quad = 0$$

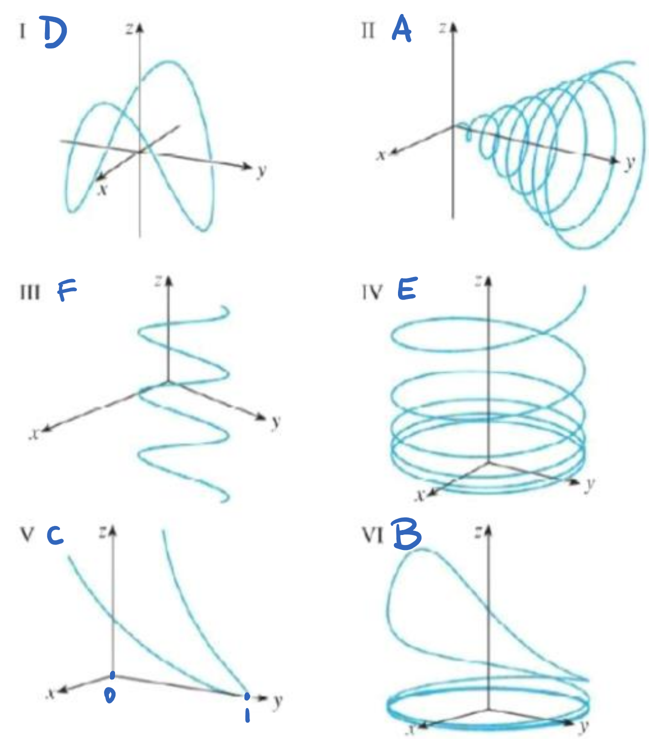
$$-4(x^2 + 2x + 1) + y^2 + 10y + 25 + 16(z^2 + 2z + 1) = -4 + 25 + 16$$

$$4 \quad (x+1)^2 + (y+5)^2 + 16(z+1)^2 = 37$$

Hyperboloid 1 sheet.

Problem 20. Match the parametric equations with the graphs (labeled I-VI)

- ~~a. $x = t\cos t$, $y = t$, $z = t\sin t$, $t \geq 0$~~
- ~~b. $x = \cos t$, $y = \sin t$, $z = \dfrac{1}{1+t^2}$~~
- ~~c. $x = t$, $y = \dfrac{1}{1+t^2}$, $z = t^2$~~
- ~~d. $x = \cos t$, $y = \sin t$, $z = \cos(2t)$~~
- ~~e. $x = \cos 8t$, $y = \sin 8t$, $z = e^{0.8t}$~~
- f. $x = \cos^2 t$, $y = \sin^2 t$, $z = t$



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