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WIR_23A_M251_H11_spaced

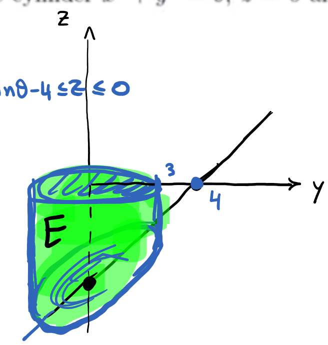
Wir 11: Chapter 16: 16.1-16.9

Problem 18. Using the The Divergence Theorem, find the flux of $\mathbf{F} = \left\langle ye^{z^2},\ ze^{x},\ 2z+8 \right\rangle$ across S, where S is the surface of the solid bounded by the cylinder $x^2+y^2=9$, $z=0$ and $z=y-4$.

$$\iiint_E \operatorname{div}\vec{F}\, dV$$

CYLINDRICAL

$$\begin{cases} y-4 \le z \le 0 \Rightarrow r\sin\theta - 4 \le z \le 0 \\ 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{cases}$$

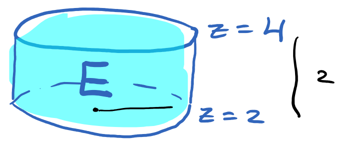


$$\operatorname{div}\vec{F} = 0 + 0 + 2$$

$$\int_0^{2\pi}\int_0^3\int_{r\sin\theta-4}^{0} 2\, dz\, r\, dr\, d\theta .$$

Problem 17. Using the The Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \left\langle 4x,\ \sin(e^z),\ \sqrt{x^3+y^2} \right\rangle$ and S is the surface ~~bounded~~ boundary of E bounded by by $x^2+y^2=4$, $z=2$, $z=4$.

Div. Thm: $\iiint_E \operatorname{div} \vec{F}\, dV =$



$= \iiint_E 4\, dV = 4 \text{Vol } E =$

$\operatorname{div} \vec{F} = 4 + 0 + 0$

$= 4\pi\, 2^2 \cdot 2 = \boxed{32\pi}$

Problem 16. Use Stokes' Theorem evaluate $\iint\limits_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = \underbrace{\langle x^2\sin(z-5), y^2, xy\rangle}$ and S is the part of the paraboliod $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward.

$C: \langle 2\cos\theta, 2\sin\theta, 5\rangle \qquad 0 \le \theta \le 2\pi$

$\int_C \vec{F}\cdot d\vec{r}$

$\vec{F}(\vec{r}(\theta)) = \langle \overbrace{4\cos^2\theta\,\sin(5-5)}^{0}, 4\sin^2\theta, 4\cos\theta\sin\theta\rangle$

$\vec{r}\,'(\theta) = \langle -2\sin\theta, \quad 2\cos\theta, \quad 0 \rangle$ DOT

$0 \quad + 8\sin^2\theta\cos\theta \quad + 0$

$\int_0^{2\pi} 8\sin^2\theta\cos\theta \, d\theta \qquad \frac{8}{3}\sin^3\theta \Big|_0^{2\pi} = 0.$

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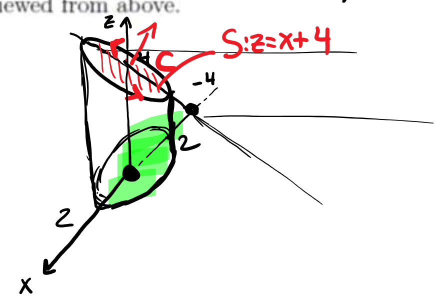
Math 251 – Spring 2023
Week-in-Review

Problem 15. Set up but do not evaluate the integral which is the result of using Stokes' Theorem to find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle 2xz, 4x^2, 5y^2 \rangle$ and C is curve of intersection of the plane $z = x+4$ and the cylinder $x^2 + y^2 = 4$, oriented counterclockwise when viewed from above.

$-x+z=4$



$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$$

$S: \vec{r}(u,v) = \langle u, v, u+4 \rangle$

$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \langle -1, 0, 1 \rangle$

$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2xz & 4x^2 & 5y^2 \end{vmatrix} =$

$= \langle 10y - 0, -(0-2x), 8x - 0 \rangle =$

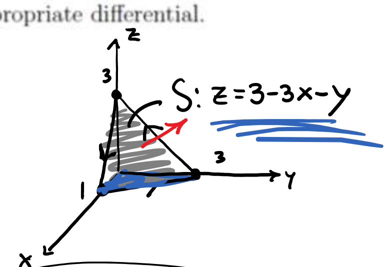
$= \langle 10y, 2x, 8x \rangle$

$\text{curl}\,\vec{F}(S) = \langle 10v, 2u, 8u \rangle \cdot \langle -1, 0, 1 \rangle = 10v + 8u$

$$\int_0^{2\pi} \int_0^2 (-10 r \sin\theta + 8 r \cos\theta)\, r\, dr\, d\theta.$$

Problem 14. Use Stokes' Theorem to set up but **not evaluate** $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle xz, 2xy, 3xy\rangle$ and where C is the boundary curve of the part of the plane $3x + y + z = 3$ in the first octant. Note: Your limits of integration must be defined with the appropriate differential.

$$\int_C F\cdot d\vec r = \iint_S \operatorname{curl}\vec F\cdot d\vec S$$



$$\operatorname{curl}\vec F = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ \partial_x & \partial_y & \partial_z \\ xz & 2xy & 3xy \end{vmatrix} =$$

$$= \boxed{\langle 3x-0, -(3y-x), 2y-0\rangle}$$

$\vec r(u,v) = \langle u, v, 3-3u-v\rangle$

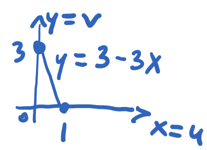
$\vec n = \langle 3,1,1\rangle$

$\vec r_u \times \vec r_v = \langle 3,1,1\rangle$

$\operatorname{curl}\vec F(\vec r(u,v)) = \langle 3u, -3v+u, 2v\rangle$ } dot

$\operatorname{curl}\vec F \cdot \vec r_u\times\vec r_v = 9u - 3v + u + 2v = 10u - v$

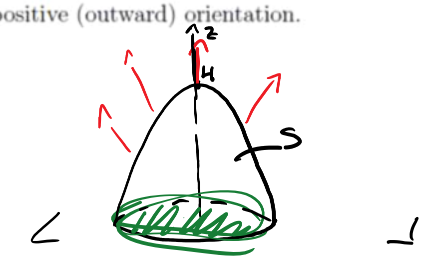
$$\int_0^1\int_0^{3-3u} (10u - v)\,dv\,du.$$



S is not closed

Problem 13. Find the flux of $\mathbf{F} = \langle\, x,\ y,\ -z \rangle$ across S, where S is the part of the paraboloid $\boxed{z = 4 - x^2 - y^2}$ that is above the xy-plane. Use the positive (outward) orientation.

$$\iint_S \vec{F} \cdot d\vec{S}$$



$$\vec{r}(u,v) = \langle u, v, 4-u^2-v^2 \rangle$$

$$r_u \times r_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} =$$

$$= \langle 2u, +2v, 1 \rangle$$

$$F(r(u,v)) = \langle u, v, -4+u^2+v^2 \rangle$$

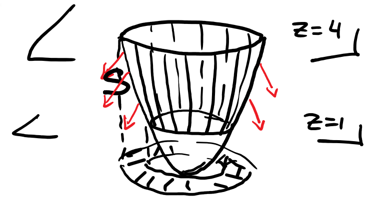
dot product $= 2u^2+2v^2-4+u^2+v^2 = \boxed{-4+3(u^2+v^2)}$

POLAR $\displaystyle\int_0^{2\pi}\int_0^2 (-4+3r^2)\, r\, dr\, d\theta = \ldots$

$$\iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, dA$$

S is not closed

Problem 12. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle y, x, z \rangle$ and S is the part of the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$.



$1 \le r \le 2$

$$S: \vec{r}(u,v) = \langle u, v, u^2+v^2 \rangle$$

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = \langle -2u, -2v, 1 \rangle$$

$$+\vec{n} = \langle 2u, 2v, -1 \rangle$$

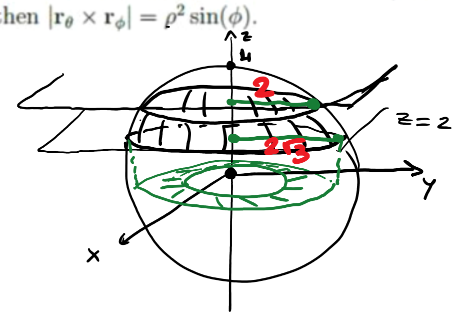
$$\vec{F}(r(u,v)) = \langle v, u, u^2+v^2 \rangle$$

dot product is $2uv + 2uv - u^2 - v^2$

$$\int_0^{2\pi}\int_1^2 (4r\cos\theta \, r\sin\theta - r^2)\, r\, dr\, d\theta$$

Problem 11. Set up but do not evaluate an integral which gives is the correct set up in order to evaluate $\iint_S yz\, dS$ where S is the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$. Note: If we parameterize the sphere $x^2 + y^2 + z^2 = \rho^2$ by $\mathbf{r}(\theta, \phi) = \langle 4\sin(\phi)\cos(\theta),\ 4\sin(\phi)\sin(\theta),\ 4\cos(\phi)\rangle$, then $|\mathbf{r}_\theta \times \mathbf{r}_\phi| = \rho^2 \sin(\phi)$.

$0 \le \theta \le 2\pi \qquad \frac{\pi}{6} \le \varphi \le \frac{\pi}{3}$

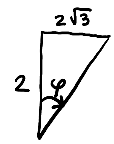


$$\iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\, dA$$

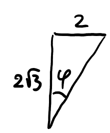
$$\int_0^{2\pi}\int_{\pi/6}^{\pi/3} 16\sin\varphi\sin\theta\cos\varphi\, 16\sin\varphi\, d\varphi\, d\theta$$

$x^2+y^2+4=16$
$x^2+y^2=12$
$r=\sqrt{12}=2\sqrt{3}$

$x^2+y^2+12=16$
$x^2+y^2=4$
$r=2$



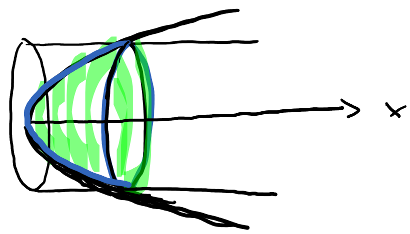
$\tan\varphi = \frac{2\sqrt{3}}{2} = \sqrt{3}$
$\varphi = \frac{\pi}{3}$



$\tan\varphi = \frac{2}{2\sqrt{3}}$
$\varphi = \frac{\pi}{6}$

Problem 10. Find the surface area of the part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

$x_y = 2y$
$x_z = 2z$



$$\iint_{D_3} \sqrt{1 + x_y^2 + x_z^2}\, dA$$

$$\sqrt{1+4y^2+4z^2}$$

$\boxed{\text{POLAR}}$

$$\int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\; r\,dr\,d\theta$$

$u = 1 + 4r^2$
$du = 8r\,dr$

$$\int_{1+0}^{1+4\cdot 9} \frac{1}{8} u^{1/2}\,du = \frac{1}{8}\,\frac{2}{3}\,u^{3/2}\Big|_1^{37} =$$

$$\frac{1}{12}\left(37^{3/2} - 1\right)$$

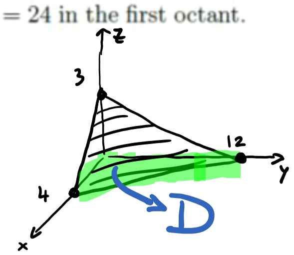
Times 2π : $\frac{\pi}{6}\left(37^{3/2} - 1\right)$

Problem 9. Find the surface area of the part of the plane $6x + 2y + 8z = 24$ in the first octant.

$$\cancel{8}z = \frac{24}{8} - \frac{6}{8}x - \frac{2}{8}y$$

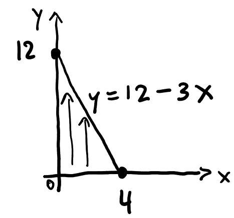
$$z = 3 - \frac{3}{4}x - \frac{1}{4}y$$

$$z_x = -\frac{3}{4} \qquad z_y = \frac{1}{4}$$



$$\boxed{\text{Area} = \iint_D \sqrt{1 + \underbrace{z_x^2}_{} + \underbrace{z_y^2}_{}}\, dA} = \iint_D |r_u \times r_v|\, dA$$

$$\int_0^4 \int_0^{12-3x} \sqrt{1 + \frac{9}{16} + \frac{1}{16}}\, dy\, dx =$$



$$\sqrt{\frac{26}{16}} \int_0^4 12 - 3x\, dx = \sqrt{\frac{13}{8}} \;\; 12x - \frac{3}{2}x^2 \Big|_0^4 =$$

$$\sqrt{\frac{13}{8}} \left(48 - \frac{3}{2}.16 - 0\right)$$

$$24\sqrt{\frac{13}{8}} = \frac{24\sqrt{13}}{2\sqrt{2}} = 12\sqrt{\frac{13}{2}}.$$

Problem 8. Given $\mathbf{F} = \langle 4xe^z, \cos(y), 2x^2e^z \rangle$ and $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $0 \le t \le \frac{\pi}{2}$.

$\vec{r}(0) = \langle 0, 0, 1 \rangle \qquad \vec{r}(\frac{\pi}{2}) = \langle 1, \frac{\pi}{2}, 0 \rangle$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 4xe^z & \cos y & 2x^2e^z \end{vmatrix} =$$

$$= \langle 0-0, -(4xe^z - 4xe^z), 0-0 \rangle$$

$f_x = 4xe^z \Rightarrow f = 2x^2e^z + g(y,z)$

$f_y = \cos y \Rightarrow f = \sin y + h(x,z)$

$f_z = 2x^2e^z \Rightarrow f = 2x^2e^z + \ell(x,y)$

$f(x,y,z) = 2x^2e^z + \sin y + C$

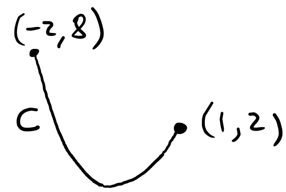
$$f(1, \tfrac{\pi}{2}, 0) - f(0,0,1) = 2 \cdot 1 \cdot 1 + \sin\tfrac{\pi}{2} - 0 - 0 = 2 + 1 = 3$$

[BY Fund. Thm. Line Integrals]

P Q

Problem 7. Find the work done by the force field $\mathbf{F} = \langle x^2,\ y^2\rangle$ in moving a particle along the arc of the parabola $y = 2x^2$ from the point $(-2, 8)$ to $(1, 2)$.

$\vec{r}(t) = \langle t, 2t^2 \rangle \qquad -2 \le t \le 1$



def. $\displaystyle\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

$\vec{F}(\vec{r}(t)) = \langle t^2, 4t^4 \rangle$

$Q_x - P_y = 0 - 0 \Rightarrow \vec{F}$ is conservative

$f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} + c$

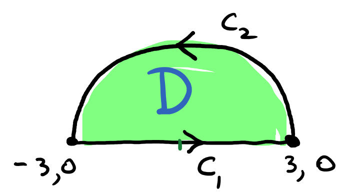
Apply Fund. Thm. Line Integrals 16.3

$$\int_C \vec{F} \cdot d\vec{r} = f(1,2) - f(-2,8) = \frac{1}{3} + \frac{8}{3} - \left(-\frac{8}{3} + \frac{8^3}{3}\right)$$

$$\frac{1}{3} + \frac{16}{3} - \frac{8^3}{3} = \frac{17 - 8^3}{3}$$

Problem 6. A particle starts at the point $(-3,0)$, moves along the x-axis to the point $(3,0)$, then along the semicircle $y=\sqrt{9-x^2}$, then back to the starting point. Find the work done on this particle by the force field $\mathbf{F}=\langle \underbrace{3x}_{P}, \underbrace{x^3+3xy^2}_{Q}\rangle$.

$$W=\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} + \int_{C_2}$$



Green's Thm : $\iint_D Q_x - P_y \, dA$

$Q_x = 3x^2+3y^2$
$P_y = 0$

In polar

$$\int_0^{\pi}\int_0^3 3r^2\, r\,dr\,d\theta$$

$$3\frac{r^4}{4}\Big|_{r=0}^{3} = 3\cdot\frac{3^4}{4} - 0$$

Answer: $\frac{3^5}{4}\pi$.

Problem 5. Evaluate $\int_C (xy)\,dx + (x-y)dy$, where C is the line segment from $(1,1)$ to $(2,0)$ and then from $(2,0)$ to $(3,5)$.

$\int_{C_1} + \int_{C_2}$

$C_1 : \vec{r}_1(t) = \langle 1+t, 1-t \rangle \quad 0 \le t \le 1$

$dx = dt \quad dy = -dt$

$C_2 : \vec{r}_2(t) = \langle 2+t, 0+5t \rangle \quad 0 \le t \le 1$

$dx = dt \quad dy = 5dt$

$C_1 : \int_0^1 (1+t)(1-t)\,dt - (\cancel{1}+t-\cancel{1}+t)\,dt =$

$1 - t^2 - 2t$

$\int_0^1 1 - t^2 - 2t\,dt = 1 - \frac{1}{3} - 1 = -\frac{1}{3}$

$C_2 : \int_0^1 (10t + 5t^2)dt + (2+t-5t)5dt = \int_0^1 5t^2 - 10t + 10dt = \frac{5}{3} - \frac{10}{2} + 10 =$

$10t + 5t^2 + 10 - 20t$

$\frac{5}{3} + 5 = \frac{20}{3}$

Answer $= -\frac{1}{3} + \frac{20}{3} = \frac{19}{3}$.

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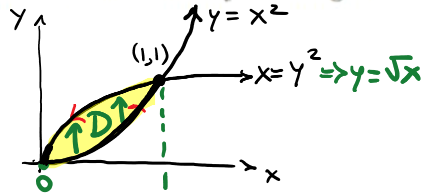
Week-in-Review

Problem 4. Find $\int_C \overbrace{(3y + 7e^{\sqrt{x}})}^{P}dx + \overbrace{(8x + 9\cos(y^2))}^{Q}dy$, where C is the boundary of the region enclosed by $y = x^2$ and $x = y^2$.



$Q_x - P_y = 8 - 3 = 5$

Green's Theorem

$$\iint_D Q_x - P_y \, dA$$

$$\iint_D 5 \, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} 5 \, dy \, dx =$$

$$\int_0^1 5(\sqrt{x} - x^2) \, dx =$$

$$= 5\left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 = 5\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{5}{3}$$

Problem 3. Evaluate $\int_C \underbrace{z dx}_{0} + (xy) dy$, where C is the line segment from $(-1, 1, 0)$ to $(1, 2, 0)$.

$\vec{r}(t) = \langle -1+2t, 1+t, 0 \rangle \quad 0 \le t \le 1$

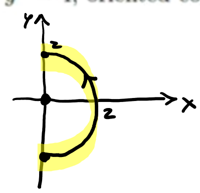
$dx = 2dt \quad dy = dt \quad dz = 0$

$$\int_0^1 0\,dx + \underbrace{(-1+2t)(1+t)}_{-1-t+2t+2t^2}\,dt = \int_0^1 (2t^2 + t - 1)\,dt = \frac{2}{3} + \frac{1}{2} - 1 = \frac{4+3-6}{6} = \boxed{\frac{1}{6}}$$

Problem 2. Find $\int_C x\,ds$, where C is the right half of the circle $x^2 + y^2 = 4$, oriented counter-clockwise.

$\vec{r}(t) = \langle 2\cos t,\ 2\sin t \rangle$

$-\frac{\pi}{2} \le t \le +\frac{\pi}{2}$



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos t \sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt =$$

$$= 4\sin t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4(1 - -1) = 8$$

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Week-in-Review

Problem 1. Evaluate $\int_C y\,ds$, where C is parameterized by $\mathbf{r}(t) = \langle t, t^3 \rangle$, $0 \le t \le 1$.

$$\int_a^b f(\vec{r}(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$\int_0^1 t^3 \sqrt{1^2 + (3t^2)^2}\, dt =$$

$$= \int_0^1 t^3 \sqrt{1+9t^4}\, dt = \int_{1+0}^{1+9} \frac{1}{36} u^{1/2}\, du =$$

$u = 1 + 9t^4$ $\quad du = 36t^3 dt$

$$= \frac{1}{\cancel{36}_{18}} \frac{\cancel{2}^{1}}{3} u^{3/2} \Big|_1^{10} = \frac{1}{54}\left(10^{3/2} - 1\right) = \frac{1}{54}\left(10\sqrt{10} - 1\right).$$