15.1 — DOUBLE INTEGRALS OVER RECTANGLES

Review

(a) We can take double integrals by using an iterated integral. If $D = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_D f(x, y)\,\mathrm{d}A = \int_a^b \int_c^d f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \int_a^b f(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

(b) The order of integration can make the problem much easier or harder.

(c) The double integral of f gives the volume under the surface $f(x, y)$.

Exercise 1

Compute the following double integrals.

(a) $\int_0^1 \int_0^1 (x + y)^2\,\mathrm{d}x\,\mathrm{d}y$

Exercise 1 continued on next page...

(b) $\displaystyle\int_1^3 \int_1^5 \frac{\ln(y)}{xy}\,\mathrm{d}y\,\mathrm{d}x$

(c) $\displaystyle\iint_R \frac{\tan\theta}{\sqrt{1-t^2}}\,\mathrm{d}A$, where $R = \{(\theta, t) : 0 \le \theta \le \pi/3, 0 \le t \le \frac{1}{2}\}$.

Exercise 2

Find the volume of a solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

Exercise 3

Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

15.2 — DOUBLE INTEGRALS OVER GENERAL REGIONS

Review

(a) If $D = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then

$$\iint_D f(x, y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

(b) If $D = \{(x, y) : g_1(y) \leq x \leq g_2(y), a \leq y \leq b\}$, then

$$\iint_D f(x, y)\,\mathrm{d}A = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

(c) The order of integration can make the problem much easier or harder.

Exercise 4

Evaluate the following double integrals.

(a) $\iint_D y\sqrt{x^2 - y^2}\,\mathrm{d}A$, where $D = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq x\}$.

Exercise 4 continued on next page...

(b) $\iint_D y^2 e^{xy}\,\mathrm{d}A$, where D is the region bounded by $y = z$, $y = 4$, and $x = 0$.

(c) $\iint_D x\cos(y)\,\mathrm{d}A$, where D is bounded by $y = 0$, $y = x^2$, and $x = 1$.

Exercise 5

Find the volume under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$.

Exercise 6

Find the volume of the region bounded by the planes $z = x$, $y = x$, $x + y = 2$ and $z = 0$.

Exercise 7

Find the volume of the region bounded by the cylinders $z = x^2$ and $y = x^2$ and the planes $z = 0$ and $y = 4$.

Exercise 8

Switch the order of integration in the following.

(a) $\int_0^2 \int_{x^2}^4 f(x, y)\,\mathrm{d}y\,\mathrm{d}x$

Exercise 8 continued on next page. . .

(b) $\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} \mathrm{d}x\,\mathrm{d}y$

Exercise 9

Evaluate the following.

(a) $\int_{0}^{1}\int_{x^2}^{1} \sqrt{y}\sin(y)\,\mathrm{d}y\,\mathrm{d}x$

Exercise 9 continued on next page...

(b) $\displaystyle\int_0^2 \int_{y/2}^1 y\cos(x^3-1)\,\mathrm{d}x\,\mathrm{d}y$

15.3 — DOUBLE INTEGRALS IN POLAR COORDS

Review

(a) If $D = \{(r,\theta) : a \le r \le b, c \le \theta \le d\}$, then

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_c^d f(r\cos\theta, r\sin\theta)\, r\,\mathrm{d}\theta\,\mathrm{d}r.$$

(b) If $D = \{(r,\theta) : g_1(\theta) \le r \le g_2(\theta), a \le r \le b\}$, then

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta.$$

(c) If $D = \{(r,\theta) : a \le r \le b, g_1(r) \le \theta \le g_2(r)\}$, then

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta)\, r\,\mathrm{d}\theta\,\mathrm{d}r.$$

(d) The order of integration can make the problem much easier or harder.

Exercise 10

Find the volume under the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant.

Exercise 11

Find the volume above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

Exercise 12

Find the volume bounded by the paraboloids $z = 6 - x^2 - y^2$ and $z = 2x^2 + 2y^2$.

Exercise 13

Compute $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\ \mathrm{d}y\,\mathrm{d}x$.