14.1 — FUNCTIONS OF SEVERAL VARIABLES

Review

(a) We plot a function of two variable above the xy-plane: $z = f(x, y)$.

(b) **Level curves** (also known as contour curves) are curves where the function $f(x, y)$ takes a specific value. Level curves satisfy the equation

$$f(x,y) = c$$

(c) Level curves are the same thing as the $z = c$ cross sections from Section 12.6.

Exercise 1

Give three real-life examples of functions of more than one variable.

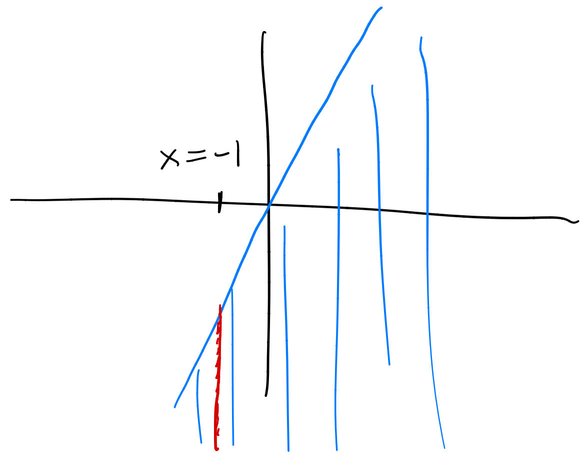
heat index = function of temperature and relative humidity.

pressure of a gas = function of its temperature and volume.

Exercise 2

Sketch the domain of the function $f(x, y) = x + \dfrac{\sqrt{2x - y}}{x + 1}$ $x \neq -1$

$$2x - y \geq 0 \Rightarrow y \leq 2x$$



The domain is the blue that is not red.

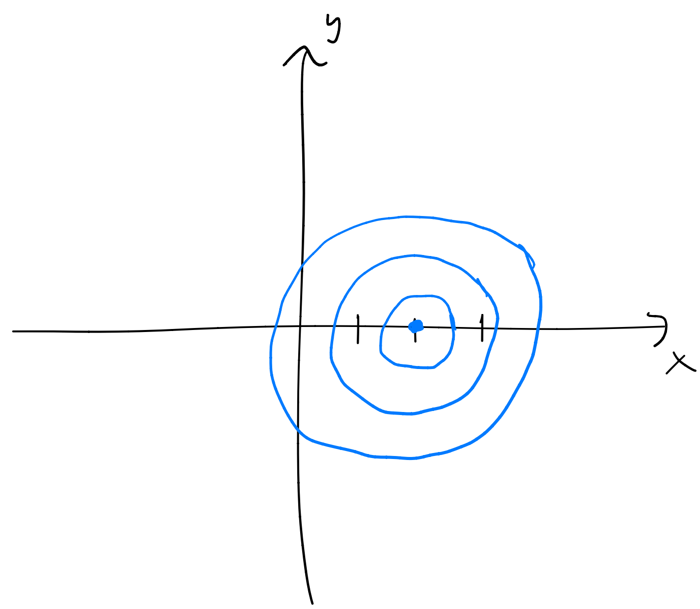
Exercise 3

Draw some level curves for the function $f(x, y) = \left((x - 2)^2 + y^2 + 1\right)^3$.

$$\left((x-2)^2 + y^2 + 1\right)^3 = c$$

$$(x-2)^2 + y^2 = \sqrt[3]{c} - 1$$

Circles centered at $(2,0)$



14.3 — PARTIAL DERIVATIVES

Review

(a) Partial derivatives are denoted in many ways. For example, the partial derivative of $f(x,y)$ with respect to x can be denoted:

$$f_x = D_x = \frac{\partial f}{\partial x} = D_1 = f_1$$

(b) To take the partial derivative with respect to a variable, pretend all other variables are constants.

(c) **Clairaut's theorem:** As long as the partial derivatives are continuous, it doesn't matter what order you take the derivatives in. For example,

$$f_{xxy}(x,y) = f_{xyx}(x,y) = f_{yxx}(x,y)$$

(d) If $z = f(x,y)$, then the partial derivative $f_x(a,b)$ can be **interpreted** as the slope of the cross section of f in the $y = b$ plane. Or equivalently, as the slope of the graph of f in the x-direction at the point (a,b).

Exercise 4

Let $f(x,y) = e^{xy}\cos(xy)$. Compute the partial derivative $\dfrac{\partial^2 f}{\partial y \partial x}$.

$$\frac{\partial f}{\partial x} = ye^{xy}\cos(xy) - ye^{xy}\sin(xy) = ye^{xy}\left(\cos(xy) - \sin(xy)\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = ye^{xy}\left(-x\sin(xy) - x\cos(xy)\right)$$
$$+ \left(yxe^{xy} + e^{xy}\right)\left(\cos(xy) - \sin(xy)\right)$$

Exercise 5

The ideal gas law states that $PV = mRT$, where P is the pressure, V is the volume, m is the mass, R is a constant, and T is the temperature.

(a) Write the pressure of the gas as function of the volume and temperature.

$$P = \frac{mRT}{V}$$

(b) Compute $\frac{\partial P}{\partial T}$ and interpret it physically.

$$\frac{\partial P}{\partial T} = \frac{mR}{V}$$

is the rate at which the pressure changes as T changes if the volume is held constant.

(c) Compute $\frac{\partial P}{\partial V}$ and interpret it physically.

$$\frac{\partial P}{\partial V} = \frac{-mRT}{V^2}$$

is the rate at which the pressure changes as V changes if the temperature is held constant.

Exercise 6

Again, consider the ideal gas law $PV = mRT$.

$$V = \frac{mRT}{P}$$

(a) Is $\dfrac{\partial P}{\partial V} = \dfrac{1}{\left(\dfrac{\partial V}{\partial P}\right)}$?

$$\frac{\partial V}{\partial P} = -\frac{mRT}{P^2} = -\frac{V}{P}.$$

$$\frac{\partial P}{\partial V} = \frac{-mRT}{V^2} = \frac{-P}{V}.$$

Yes.

(b) What do you think $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P}$ is equal to? Compute it and find out.

I think it's equal to 1.

$$\frac{\partial P}{\partial V} = \frac{-mRT}{V^2}$$

$$V = \frac{mRT}{P} \qquad \frac{\partial V}{\partial T} = \frac{mR}{P}$$

$$T = \frac{PV}{mR} \qquad \frac{\partial T}{\partial P} = \frac{V}{mR}$$

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = \frac{-mRT}{V^{\cancel{2}}}\,\frac{\cancel{mR}}{P}\,\frac{\cancel{V}}{\cancel{mR}}$$

$$= \frac{-mRT}{PV} = -1.$$

by the ideal gas

14.4 — TANGENT PLANES AND APPROXIMATIONS

Review

(a) The formula for the **tangent plane** to the graph of $f(x, y)$ at the point (a, b, c) is

$$f_x(a,b)(x-a)+f_y(a,b)(y-b)-(z-c)=0$$

(b) The **total differential** dz for a function $z = f(x, y)$ is defined as

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

(c) Using the total differential, one can obtain **linear approximations** to the the function.

Exercise 7

Find the tangent plane to the graph of $f(x, y) = xy^3$ at $(-3, 1, -3)$.

$$f_x(x,y) = y^3 \qquad f_x(-3,1) = 1$$

$$f_y(x,y) = 3xy^2 \qquad f_y(-3,1) = 3(-3)\cdot 1^2 = -9$$

$$\boxed{(x+3)-9(y-1)-(z+3)=0}$$

Exercise 8

Find the total differential for the function $w = xe^z + x\sin(yz)$.

$$\frac{\partial w}{\partial x} = e^z + \sin(yz)$$

$$\frac{\partial w}{\partial y} = xz\cos(yz)$$

$$\frac{\partial w}{\partial z} = xe^z + xy\cos(yz)$$

$$dw = \left(e^z + \sin(yz)\right)dx + xz\cos(yz)dy + \left(xe^z + xy\cos(yz)\right)dz$$

Exercise 9

Using the total differential, approximate $g(x, y, z) = xe^z + x\sin(yz)$ at $(3.2, -0.9, 0.2)$.

Nearby point: $(3, -1, 0)$.

$$dw = \left(e^0 + \sin(0)\right)(0.2) + (3)(0)\cos(0)(0.1) + \left(3e^0 + (3)(-1)\cos(0)\right)(0.2)$$

$$= 0.2$$

$$g(3.2, -0.9, 0.2) \approx g(3, -1, 0) + dw$$

$$= 3e^0 + 3\sin(0) + 0.2$$

$$= \boxed{3.2}$$

14.5 — THE CHAIN RULE

Review

(a) Procedure for using the chain rule with multivariable functions:

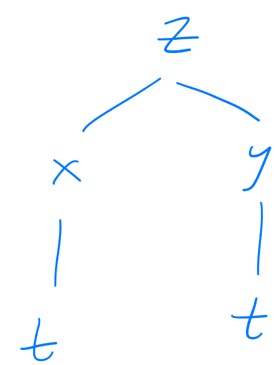
(i) Draw the tree diagram of the dependence of the variables.

(ii) Write the partial derivatives on the branches of the tree.

(iii) Add up all the branches that reach to the variable you want to take the derivative with respect to.

Exercise 10

$z = g(x, y)$, $x = \tan(t)$, $y = t^3$. Find $\dfrac{\partial g}{\partial t}$.

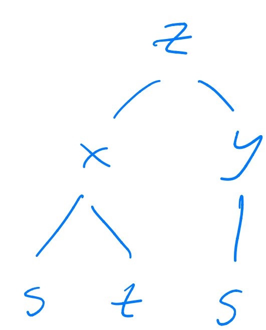


$$\frac{\partial g}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= g_x(x,y)\sec^2(t) + g_y(x,y)\cdot 3t^2$$

Exercise 11

$z = f(x, y)$, $x = st^2$, $y = \cos(s)$. Find $\dfrac{\partial f}{\partial s}$.

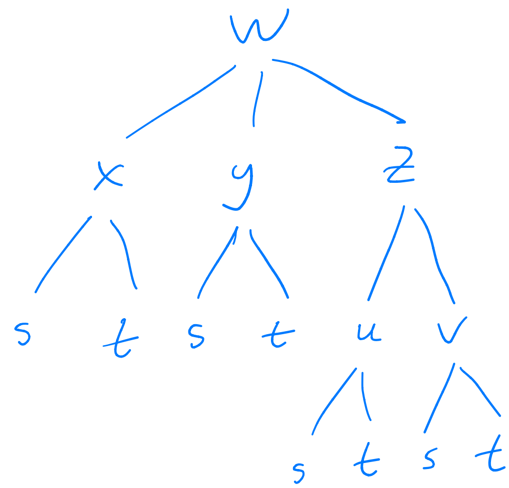


$$\frac{\partial f}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{dy}{ds}$$

$$= f_x(x,y)\,t^2 + f_y(x,y)(-\sin(s))$$

Exercise 12

$w = \cos(xyz)$, $x = st$, $y = \frac{s}{t}$, $z = e^{uv}$, $u = t\cos(s)$, $v = st^2$. Find $\dfrac{\partial w}{\partial t}$.



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}\frac{\partial v}{\partial t}$$

$$= -yz\sin(xyz)s - xz\sin(xyz)(-st^{-2}) - xy\sin(xyz)ve^{uv}\cos(s) - xy\sin(xyz)ue^{uv}\cdot 2st$$