EXAM 1 REVIEW

Exercise 1

Describe the following regions of $\mathbb{R}^3$ in words.

(a) $z > 0$

(b) $x = y$

(c) $x^2 + y^2 = 4$

(d) $(x-1)^2 + (y-2)^2 + (z+1)^2 \leq 9$

(e) $1 \leq z^2 + x^2 \leq 9$

(a) everything above the xy-plane

(b) the plane $x=y$

(c) cylinder with radius 2 centered at the origin that extends infinitely in the z-direction.

(d) filled in ball of radius 3 centered at $(1,2,-1)$.

(e) filled in cylinder of radius 3 with the middle part of radius 1 cut out.

Exercise 2

What is the intersection of the following regions in $\mathbb{R}^3$?

(a) $x \geq 0$ and $x^2 + y^2 + z^2 \leq 4$

(b) $x^2 + z^2 = 1$ and $y = 2$

(c) $x^2 + z^2 = 4$ and $z = 1$

(d) $1 \leq x^2 + y^2 + z^2 \leq 9$ and $y = 2$

(a) half of the filled in ball centered at the origin with radius 2.

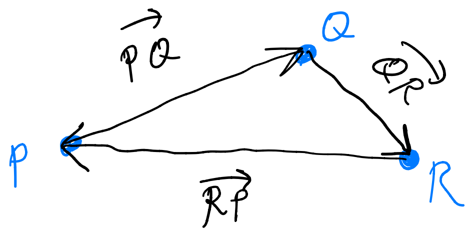
(b) circle of radius 1.

(c) two lines

(d) a filled in circle

Exercise 3

P, Q, and R form a triangle. What is $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP}$?



They cancel each other out, so you get $\vec{0}$.

Exercise 4

Let $\mathbf{u} = \langle 3, -1, 2 \rangle$. Find a vector $\mathbf{v}$ such that $\mathbf{v}$ goes in the direction of $\langle 1, 2, -2 \rangle$ and $\text{comp}_{\mathbf{u}}\mathbf{v} = -4$.

$\vec{v} = c\langle 1, 2, -2 \rangle$ for some c. (we want to find c.)

$$\text{comp}_{\vec{u}}\vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} = \frac{3c-2c-4c}{\sqrt{3^2+(-1)^2+2^2}} = \frac{-3c}{\sqrt{14}} = -4.$$

$$\Rightarrow c = \frac{4\sqrt{14}}{3}$$

$$\boxed{\vec{v} = \left\langle \frac{4\sqrt{14}}{3}, \frac{8\sqrt{14}}{3}, \frac{-8\sqrt{14}}{3} \right\rangle}$$

Exercise 5

Is the triangle formed by the vertices $A(1, 2, 3)$, $B(5, 1, 6)$, $C(3, 4, 1)$ a right triangle?

Idea: Take the dot product of the sides. If a dot product is 0, then the sides are perpendicular (i.e., it is a right triangle).

$\overrightarrow{AB} = \langle 4, -1, 3 \rangle$

$\overrightarrow{BC} = \langle -2, 3, -5 \rangle$

$\overrightarrow{AC} = \langle 2, 2, -2 \rangle$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = 4(-2) - 1(3) + 3(-5) = -26.$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 4(2) - 1(2) + 3(-2) = \underline{\underline{0}}.$

Yes, it is a right triangle.

Exercise 6

Find the vector projection of $\overrightarrow{AB}$ onto $\overrightarrow{BC}$.

$$\text{proj}_{\overrightarrow{BC}} \overrightarrow{AC} = \frac{\overrightarrow{AB} \cdot \overrightarrow{BC}}{|\overrightarrow{BC}|^2} \overrightarrow{BC}$$

$$= \frac{-26}{(-2)^2 + 3^2 + (-5)^2} \langle -2, 3, -5 \rangle$$

$$\boxed{= \frac{-26}{38} \langle -2, 3, -5 \rangle.}$$

Exercise 7

Find a unit vector perpendicular to the plane containing the point $(-2, 5, 2)$ and the line $L(t) = \langle 2t, 3+t, 1-2t\rangle$.

The line contains the point $(0,3,1)$ and goes in the direction $\langle 2,1,-2\rangle$. So, subtracting the points, we get another vector in the plane:

$\langle -2-0, 5-3, 2-1\rangle = \langle -2, 2, 1\rangle$. Take their cross product to find a vector $\perp$ to the plane:

$\langle 2,1,-2\rangle \times \langle -2,2,1\rangle = \langle 5,2,6\rangle$. Then divide by the length to get a unit vector: $\boxed{\frac{1}{\sqrt{5^2+2^2+6^2}}\langle 5,2,6\rangle.}$

Exercise 8

Let V be the parallelepiped whose edges all have length 2. One side of V lies in the xy-plane. The angle between the edges that lie in the xy-plane is $45°$. An edge of V that is not in the xy-plane makes a $30°$ angle with the z-axis. What is the volume of V?

$V = |\vec{a}\cdot(\vec{b}\times\vec{c})|$. Let $\vec{b}$ and $\vec{c}$ be the vectors in the xy-plane. Then, $|\vec{b}\times\vec{c}| = |\vec{b}||\vec{c}|\sin\theta$

$$= 2\cdot 2\cdot \sin(45°) = 2\sqrt{2}.$$

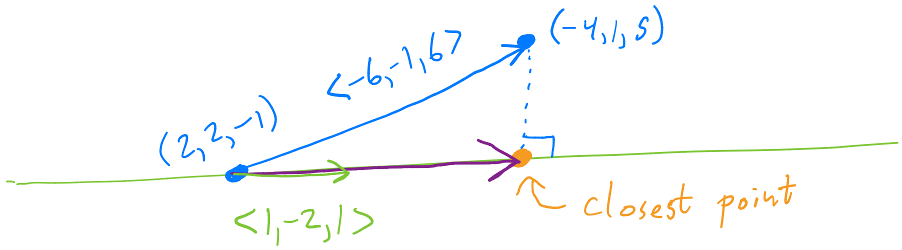
By the right hand rule, the direction of $\vec{b}\times\vec{c}$ is either in the z or $-z$ direction. Either way,

$$|\vec{a}\cdot(\vec{b}\times\vec{c})| = |\vec{a}|\,|\vec{b}\times\vec{c}|\underbrace{|\cos(\overset{= 30° \text{ or } 150°}{\theta})|}_{=\frac{\sqrt{3}}{2}}$$

$$= 2\cdot 2\sqrt{2}\cdot\frac{\sqrt{3}}{2} = \boxed{2\sqrt{6}.}$$

Exercise 9

Find the point on the line $L(t) = \langle 2+t, 2-2t, -1+t \rangle$ that is closest to the point $(-4, 1, 5)$.



$$\text{proj}_{\langle 1,-2,1\rangle} \langle -6,-1,6\rangle = \frac{\langle -6,-1,6\rangle \cdot \langle 1,-2,1\rangle}{1^2+(-2)^2+1^2} \langle 1,-2,1\rangle = \frac{2}{6}\langle 1,-2,1\rangle$$

$$= \left\langle \frac{1}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

$$(2,2,-1) + \left\langle \frac{1}{3}, \frac{-2}{3}, \frac{1}{3} \right\rangle = \boxed{\left(\frac{7}{3}, \frac{4}{3}, \frac{-2}{3}\right).}$$

Exercise 10

What is the domain of $\mathbf{r}(t) = \langle \sqrt{2t+4}, \ln(3-t), (1-t)^{-1} \rangle$?

$\sqrt{2t+4}$ is defined for $2t+4 \geq 0 \Rightarrow t \geq -2$.

$\ln(3-t)$ is defined for $3-t > 0 \Rightarrow t < 3$.

$(1-t)^{-1}$ is defined for $t \neq 1$.

The domain of $\vec{r}(t)$ is where all the components are defined, so the domain of $\vec{r}(t)$ is

$$\boxed{[-2, 1) \cup (1, 3)}$$

Exercise 11

Find parametric and symmetric equations for a line that is perpendicular to the plane $3x - 7y + 4z = 8$ and passes through the point $(5, 1, -4)$.

Normal vector to the plane: $< 3, -7, 4 >$.

The line must go in this direction, so its parametric equation is

$$\boxed{L(t) = < 5, 1, -4 > + t < 3, -7, 4 >}$$

To find the symmetric equations, solve for t and set them equal to each other:

$$x = 5 + 3t \Rightarrow t = \frac{x-5}{3}$$

$$y = 1 - 7t \Rightarrow t = \frac{y-1}{-7}$$

$$z = -4 + 4t \Rightarrow t = \frac{z+4}{4}$$

$$\boxed{\frac{x-5}{3} = \frac{y-1}{-7} = \frac{z+4}{4}}$$

Exercise 12

Determine if the following pair of lines is intersecting, parallel, or skew: $L_1(t) = \langle 1 + 2t, -2 - t, 3 + 2t \rangle$ and $L_2(t) = \langle 1 + t, -2 + 3t, 1 + 2t \rangle$.

L_1 goes in the direction of $< 2, -1, 2 >$.

L_2 goes in the direction of $< 1, 3, 2 >$.

Not parallel.

If they intersect, then there is t and s such that $L_1(t) = L_2(s)$.

$$1 + 2t = 1 + s \Rightarrow s = 2t$$

$$-2 - t = -2 + 3t \Rightarrow t = 0 \Rightarrow s = 0$$

$L_1(0) = < 1, -2, 3 >$

$L_2(0) = < 1, -2, 1 >$

not equal, so they do not intersect.

$\boxed{L_1 \text{ and } L_2 \text{ are skew}}$

Exercise 13

Find a plane whose intercepts with the x, y, and z axes are 3, 7, and -2, respectively.

$(3,0,0)$, $(0,7,0)$, and $(0,0,-2)$ are on the plane.

$\Rightarrow \langle 3,-7,0\rangle$ and $\langle 3,0,2\rangle$ are in the plane.

$\Rightarrow \langle 3,-7,0\rangle \times \langle 3,0,2\rangle = \langle -14,-6,21\rangle$ is $\perp$ to the plane.

Therefore, the plane is

$$\boxed{-14(x-3) - 6(y-0) + 21(z-0) = 0.}$$

Exercise 14

Find the intersection between the plane $x + y = z$ and the plane $3x - 2y - 2z = 5$.

The normal vectors for these planes are $\langle 1,1,-1\rangle$ and $\langle 3,-2,-2\rangle$. The line must be $\perp$ to both these normal vectors since the line lies in both planes. So, we can find the direction of the line by taking the cross product of these normal vectors:

$\langle 1,1,-1\rangle \times \langle 3,-2,-2\rangle = \langle -4,1,-5\rangle.$

Now, find a point on the line by plugging $z = x+y$ into $3x-2y-2z=5$:

$3x-2y-2(x+y)=5 \Rightarrow x = 5+4y.$ Choose $y=0$. Then, $x=5$ and $z=5$.

Finally, plug this into the equation for a line:

$$\boxed{L(t) = \langle 5,0,5\rangle + t\langle -4,1,-5\rangle}$$

Exercise 15

Draw the traces of the equation $x^2 - 3y^2 + z^2 = 4$. What shape is it?

$z = c$ traces:

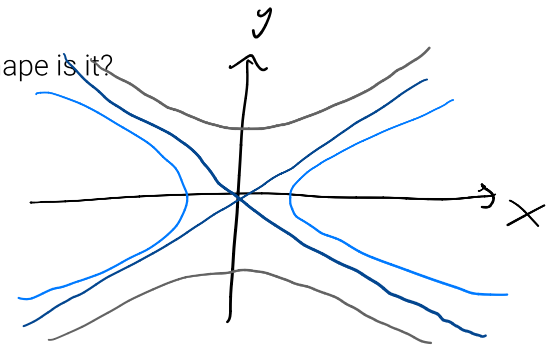
$$x^2 - 3y^2 = 4 - c^2 \quad \text{hyperbolas}$$

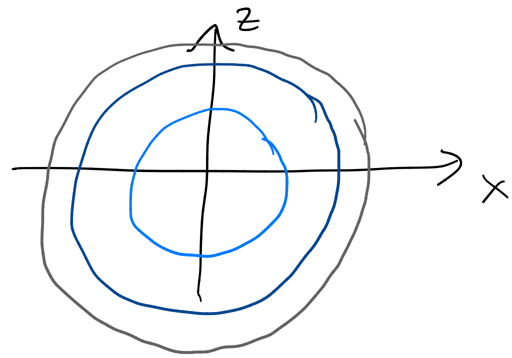
$y = c$ traces:

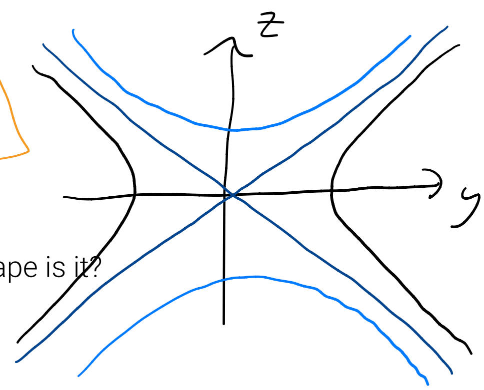
$$x^2 + z^2 = 4 + 3c^2 \quad \text{circles}$$

$x = c$ traces:

$$-3y^2 + z^2 = 4 - c^2 \quad \text{hyperbolas}$$



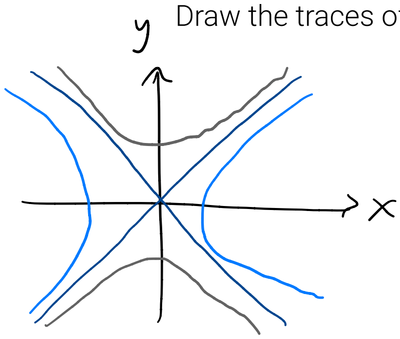


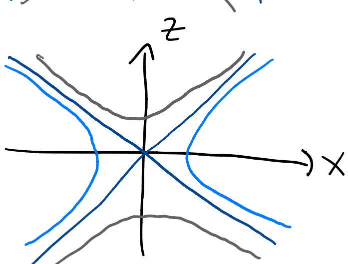


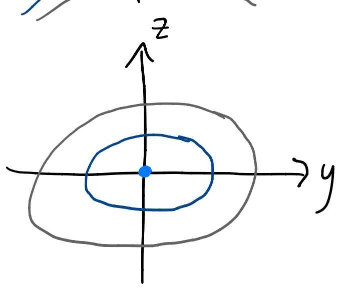
hyperboloid of one sheet

Exercise 16

Draw the traces of the equation $3x^2 - y^2 - 2z^2 = 0$. What shape is it?







$z = c$ traces:

$$3x^2 - y^2 = 2c^2 \quad \text{hyperbolas}$$

$y = c$ traces:

$$3x^2 - 2z^2 = c^2 \quad \text{hyperbolas}$$

$x = c$ traces:

$$-y^2 - 2z^2 = -3c^2 \quad \text{ellipses}$$

Cone

Exercise 17

Find the intersection between the curve $\mathbf{r}(t) = \langle t^2, \cos(t), \sin(t)\rangle$ and the surface $3x^2+2y^2+2z^2 = 5$.

Plug $x=t^2$, $y=\cos(t)$, $z=\sin(t)$ into the surface equation.

$$3(t^2)^2+2\cos^2(t)+2\sin^2(t)=5$$

$$3t^4+2=5$$

$$3t^4=-3$$

$$t^4=-1$$

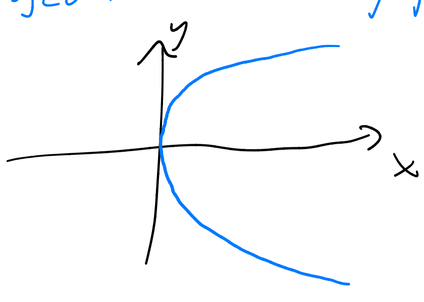
$\Rightarrow t=\pm 1$, so $\vec{r}(t)$ intersects the surface at

$\vec{r}(-1) = \underline{(1, \cos(-1), \sin(-1))}$ and $\vec{r}(1) = \underline{(1, \cos(1), \sin(1))}$.

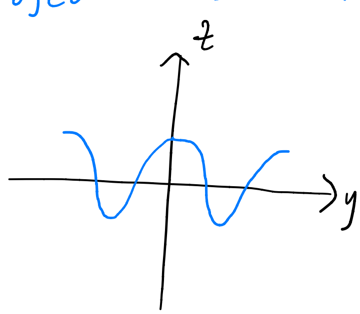
Exercise 18

Draw the projection of $\mathbf{r}(t) = \langle t^2, t, \cos(t)\rangle$ onto the xy and yz planes.

Projection onto xy-plane: Ignore the z-coordinate.



Projection onto the yz-plane: Ignore the x-coordinate.



Exercise 19

Find the line tangent to $\mathbf{r}(t) = \langle t^2, t, \cos(\pi t)\rangle$ at the point $(4, -2, 1)$.

$(4,-2,1)$ corresponds to $t=-2$. i.e., $\vec{r}(-2) = \langle 4,-2,1\rangle$.

$$\vec{r}\,'(t) = \langle 2t, 1, -\pi\sin(\pi t)\rangle.$$

$$\vec{r}\,'(-2) = \langle -4, 1, -\pi\sin(-2\pi)\rangle$$

$$= \langle -4, 1, 0\rangle.$$

$$\boxed{L(t) = \langle 4,-2,1\rangle + t\langle -4,1,0\rangle.}$$

Exercise 20

What is $\displaystyle\int_1^2 \left(\sin(\pi t)\mathbf{i} + e^{2t}\mathbf{j} - 7\mathbf{k}\right) \mathrm{d}t$?

Just integrate each component separately.

$$\int_1^2 \sin(\pi t)\,\vec{i} + e^{2t}\,\vec{j} - 7\vec{k}\,dt$$

$$= \int_1^2 \sin(\pi t)\,dt\,\vec{i} + \int_1^2 e^{2t}\,dt\,\vec{j} - 7\int_1^2 dt\,\vec{k}$$

$$= -\frac{1}{\pi}\cos(\pi t)\Big|_1^2\,\vec{i} + \frac{1}{2}e^{2t}\Big|_1^2\,\vec{j} - 7\vec{k}$$

$$= -\frac{1}{\pi}\left(\cos(2\pi) - \cos(\pi)\right)\vec{i} + \frac{1}{2}\left(e^4 - e^2\right)\vec{j} - 7\vec{k}$$

$$\boxed{= \frac{-2}{\pi}\,\vec{i} + \frac{1}{2}(e^4 - e^2)\,\vec{j} - 7\vec{k}}$$

Exercise 21

Find the length of the curve $\mathbf{r}(t) = \langle \sin(\pi t), 3t, \cos(\pi t) \rangle$ from $(0, 0, 1)$ to $(0, 6, 1)$.

$(0,0,1)$ corresponds to $t=0$.

$(0,6,1)$ corresponds to $t=2$.

$$\vec{r}\,'(t) = \langle \pi \cos(\pi t), 3, -\pi \sin(\pi t) \rangle$$

$$L = \int_0^2 |\vec{r}\,'(t)|\, dt$$

$$= \int_0^2 \sqrt{(\pi \cos(\pi t))^2 + 3^2 + (-\pi \sin(\pi t))^2}\, dt$$

$$= \int_0^2 \sqrt{\pi^2 + 9}\, dt$$

$$= \boxed{2\sqrt{\pi^2+9}}.$$