12.1 — 3D SPACE

Review

(a) The xyz axes should satisfy the right hand rule.

(b) In $\mathbb{R}^3$, the xz-coordinate plane is the plane containing the x and z axes. How can this plane be written in terms of an equation?

$y = 0$

(c) To find a sphere, you need its center and radius. If (c_1, c_2, c_3) is the center and the radius is r, what is the standard formula for the equation of a sphere?

$$(x-c_1)^2 + (y-c_2)^2 + (z-c_3)^2 = r^2$$

(d) What is the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2)?

$$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$$

Exercise 1

Find an equation for the sphere with center at $(2, 1, -3)$ that just touches the plane $y = 5$.

centered on $y=1$ and just touches $y=5 \Rightarrow r=4$.

$$(x-2)^2 + (y-1)^2 + (z+3)^2 = 4^2.$$

Exercise 2

Let S be the sphere given by the equation $x^2 - 2x + y^2 + 8y + z^2 + 6z = 10$. Find an equation for the sphere centered at $(3, 2, 6)$ that just barely touches S.

$$x^2 - 2x + 1 + y^2 + 8y + 16 + z^2 + 6z + 9 = 10 + 1 + 16 + 9$$

$$(x-1)^2 + (y+4)^2 + (z+3)^2 = 36 = 6^2.$$

S has center $(1, -4, -3)$ and radius 6.

distance between the centers of the spheres is

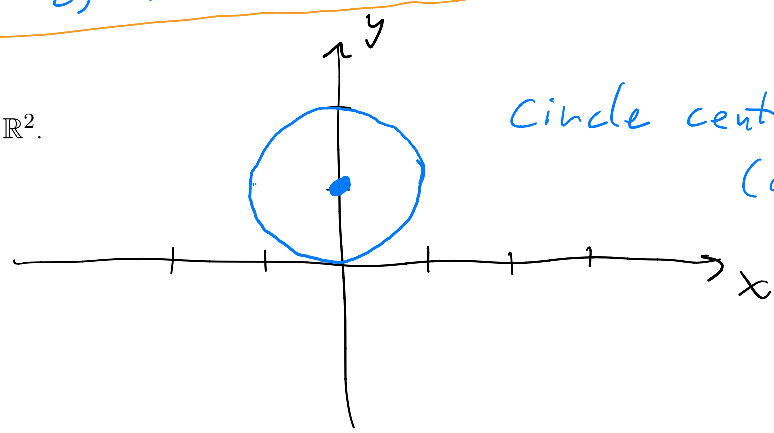
$$\sqrt{(3-1)^2 + (2-{-4})^2 + (6-{-3})^2} = 11.$$

So, the other sphere must have radius 5.

$$\boxed{(x-3)^2 + (y-2)^2 + (z-6)^2 = 5^2}$$

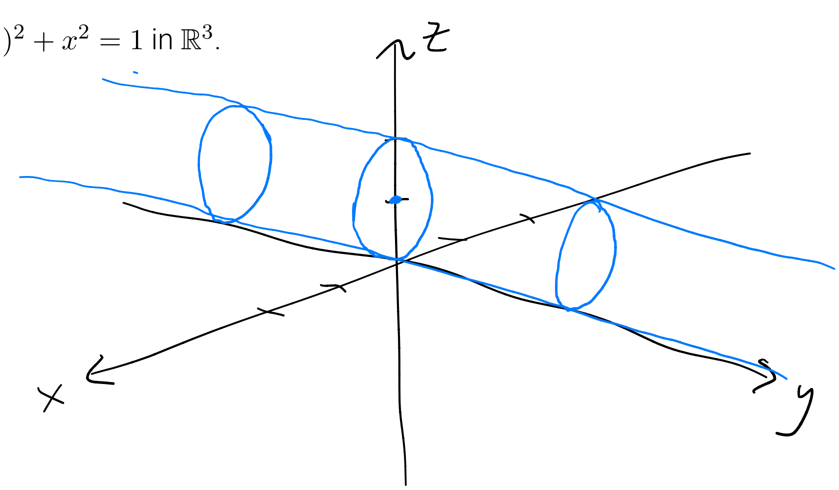
Exercise 3

(a) Plot $(y-1)^2 + x^2 = 1$ in $\mathbb{R}^2$.



Circle centered at $(0, 1)$.

(b) Plot $(z-1)^2 + x^2 = 1$ in $\mathbb{R}^3$.



cylinder

Exercise 4

In words, state what the following regions represent in $\mathbb{R}^3$.

(a) $x^2 + y^2 + z^2 \le 1$ and $z > 0$.

upper half of the unit ball centered at the origin

(b) $(x-4)^2 + (z+2)^2 = 7$.

cylinder with radius $\sqrt{7}$ that is infinitely long in the y-direction. The circular slices are centered at $(4, y, -2)$.

(c) $(x-4)^2 + (z+2)^2 = 7$ and $y = 4$.

Circle in the plane $y=4$ with radius $\sqrt{7}$ and centered at $(4, 4, -2)$.

Exercise 5

Let R be the region in $\mathbb{R}^3$ defined by $1 \le (x-1)^2 + (y+2)^2 + z^2 \le 9$. Let P be a plane. What are the possible shapes of $R \cap P$ (R intersected with P)?

1. $\emptyset$ (i.e., they might not intersect)
2. a single point (if the plane is tangent to the outer surface of R)
3. a filled in circle (if the plane cuts through the outer surface, but not through the inner surface)
4. a filled ring  (if the plane cuts through the outer and the inner surfaces.

12.2 — VECTORS

Review

(a) A vector has a direction and a magnitude.

(b) If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then the length of $\mathbf{v}$ is

$$\sqrt{V_1^2 + V_2^2 + V_3^2}$$

(c) A unit vector is a vector with length 1.

(d) $\mathbf{i} = \langle 1, 0, 0 \rangle$ is the unit vector in the x-direction.

$\mathbf{j} = \langle 0, 1, 0 \rangle$ is the unit vector in the y-direction.

$\mathbf{k} = \langle 0, 0, 1 \rangle$ is the unit vector in the z-direction.

Exercise 6

An airplane is currently (with respect to the ground) flying 200mph west, 100mph north, and 30mph up.

(a) How fast is the airplane going (with respect to the ground)?

The speed is the length of the velocity vector:

$$\sqrt{200^2 + 100^2 + 30^2} = \sqrt{50900} \text{ mph}$$

(b) Assuming the airplane is oriented in its direction of travel, at what angle is the nose pointed up?

$$\theta = \sin^{-1}\left(\frac{30}{\sqrt{50900}}\right).$$

Exercise 7

Let $A = (5, 2, 7)$ and $B = (-2, 7, -3)$. Find the unit vector that points in the direction from A to B.

First find some vector that points from A to B:

$$\vec{v} = (-2-5, 7-2, -3-7) = (-7, 5, -10).$$

Now turn $\vec{v}$ into a unit vector by dividing by $|\vec{v}|$.

$$|\vec{v}| = \sqrt{(-7)^2 + 5^2 + (-10)^2} = \sqrt{174}.$$

$$\frac{1}{|\vec{v}|}\vec{v} = \boxed{\left(\frac{-7}{\sqrt{174}}, \frac{5}{\sqrt{174}}, \frac{-10}{\sqrt{174}}\right)}$$

12.3 – THE DOT PRODUCT

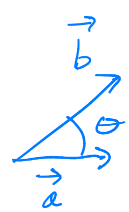
Review

(a) The dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

which is also equal to

$$|\vec{a}|\,|\vec{b}| \cos\theta$$



(b) Two vectors are perpendicular (or orthogonal) if the angle between them is $\pi/2$ (i.e., $90°$).

(c) Which vectors is the zero vector $\mathbf{0}$ orthogonal to?

$\vec{0}$ is perpendicular to all vectors.

(d) Two vectors are orthogonal if and only if their dot product is $\underline{0}$.

Exercise 8

Let $\mathbf{v} = a\mathbf{i} + 3\mathbf{j}$. For which values of a is $\mathbf{v}$ orthogonal to $\langle 1, 2, 5\rangle$?

$\vec{v} \cdot \langle 1,2,5\rangle = a+6 = 0 \iff \boxed{a=-6}$

Exercise 9

Suppose $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 2$.

(a) If $\mathbf{a} \cdot \mathbf{b} = 6$, what do we know about the orientations of $\mathbf{a}$ and $\mathbf{b}$?

$6 = \vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = 6\cos\theta$

$\Rightarrow \cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow \vec{a}$ and $\vec{b}$ are pointing in the same direction

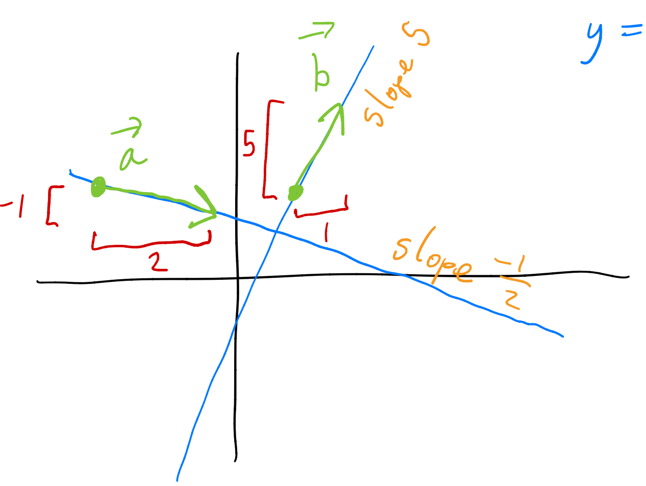
(b) If $\mathbf{a} \cdot \mathbf{b} = -6$, what do we know about the orientations of $\mathbf{a}$ and $\mathbf{b}$?

$-6 = \vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = 6\cos\theta$

$\Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi \Rightarrow \vec{a}$ and $\vec{b}$ are pointing in opposite directions

Exercise 10

Find the angle between the lines $x + 2y = 7$ and $5x - y = 2$.



$y = -\frac{x}{2} + \frac{7}{2}$ $\qquad y = 5x - 2$

$\vec{a} = \langle 2, -1\rangle$

$\vec{b} = \langle 1, 5\rangle$

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

$2 \cdot 1 + (-1)(5) = \sqrt{2^2 + (-1)^2}\sqrt{1^2 + 5^2} \cos\theta$

$-3 = \sqrt{130} \cos\theta$

$\boxed{\theta = \cos^{-1}\left(\frac{-3}{\sqrt{130}}\right)}$