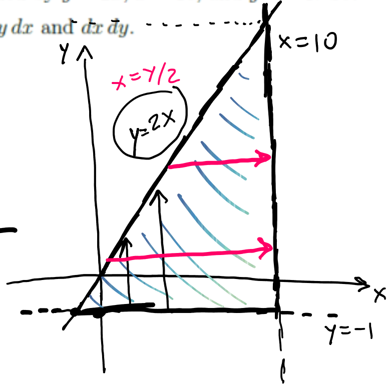
Wir 9: Exam 3 Review

Sections 15.1-15.~~4~~3, 15.6-15.9

Problem 1. Let R be the region in the xy–plane bounded by $y = 2x$, $x = 10$, and $y = -1$. Set up but do not evaluate $\int\int_R (x^2 + y^2)\, dA$ in the order $dy\, dx$ and $dx\, dy$.

$dy\,dx \begin{cases} -\frac{1}{2} \le x \le 10 \\ -1 \le y \le 2x \end{cases}$

$$\int_{-1/2}^{10}\int_{-1}^{2x} (x^2+y^2)\,dy\,dx$$



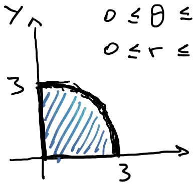
$dx\,dy \begin{cases} -1 \le y \le 20 \\ \frac{y}{2} \le x \le 10 \end{cases}$

$$\int_{-1}^{20}\int_{y/2}^{10} (x^2+y^2)\,dx\,dy$$

$2x = -1$
$x = -1/2$

$x^2+y^2=r^2$

Problem 2. Evaluate $\int_0^3 \int_0^{\sqrt{9-x^2}} e^{-x^2-y^2}\, dy\, dx$



$0 \le \theta \le \frac{\pi}{2}$
$0 \le r \le 3$

Polar Coordinates:

$$\int_0^{\pi/2}\int_0^3 e^{-r^2}\, r\,dr\,d\theta =$$

$u = -r^2$
$du = -2r\,dr$

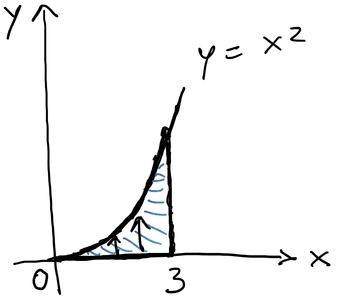
$y = \sqrt{9-x^2}$
$y^2 = 9 - x^2$

$$= \frac{\pi}{2}\int_0^{-9} -\frac{1}{2}e^u\,du =$$

$$= -\frac{\pi}{4}\left[e^u\right]_0^{-9} = -\frac{\pi}{4}\left(e^{-9}-1\right) = \frac{\pi}{4}\left(1-\frac{1}{e^9}\right)$$

Problem 3. Let D be the region bounded by $y = 0$, $y = x^2$, and $x = 3$. Find $\iint_D 3x\cos y\,dA$.

$$\int_0^3\int_0^{x^2} 3x\cos y\,dy\,dx$$



$$3x\left[\sin y\right]_{y=0}^{y=x^2}$$

$$3\ x\left\{\sin(x^2)\ -0\right\}$$

$u = x^2$

$du = 2x\,dx$

$$\int_0^3 3\,x\sin x^2\,dx = \int_0^9 3\frac{1}{2}\sin u\,du =$$

$$= \frac{3}{2}\left[-\cos u\right]_0^9 = \frac{3}{2}\left(-\cos 9 - -1\right) = \frac{3}{2}\left(1-\cos 9\right).$$

$y = x/3$

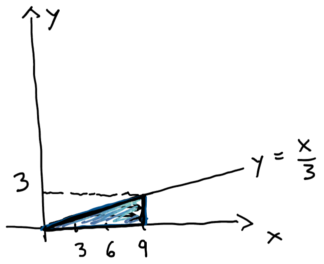
$3y \le x \le 9$

$0 \le y \le 3$

Problem 4. Compute $\int_0^3\int_{3y}^9 7e^{x^2}\,dx\,dy$.

Switch to dy dx

$$\int_0^9\int_0^{x/3} 7e^{x^2}\,dy\,dx$$



$$\begin{cases} 0 \le x \le 9 \\ 0 \le y \le \frac{x}{3} \end{cases}$$

$$7e^{x^2}\left(\frac{x}{3}-0\right)$$

$x^2 = u$

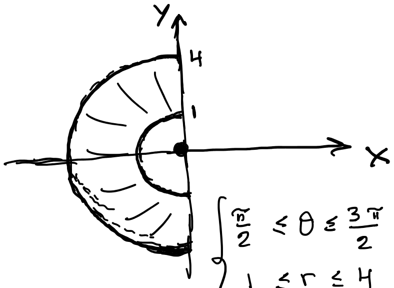
$du = 2x\,dx$

$$\int_0^9 \frac{7}{3}\,x\,e^{x^2}\,dx = \int_0^{81}\frac{7}{3}\frac{1}{2}e^u\,du =$$

$$= \frac{7}{6}\,e^u\Big|_0^{81} = \frac{7}{6}\left(e^{81}-1\right)$$

Problem 5. Let R be the region that lies to the left of the y–axis between the circles $x^2+y^2=1$ and $x^2+y^2=16$. Find $\int\int_R 5(x+y)dA$.

Polar Coordinates:



$$\begin{cases} \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \\ 1 \le r \le 4 \end{cases}$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{1}^{4} 5\left(r\cos\theta + r\sin\theta\right) r \, dr \, d\theta =$$

$$5\int_{1}^{4} r^2 \left(\cos\theta + \sin\theta\right) dr =$$

$$= 5\left(\cos\theta + \sin\theta\right)\frac{1}{3}r^3 \Big|_{1}^{4} =$$

$$\frac{16}{4}\over{64}$$

$$= \int \frac{5}{3}\left(\cos\theta + \sin\theta\right)\left(4^3-1\right) d\theta$$

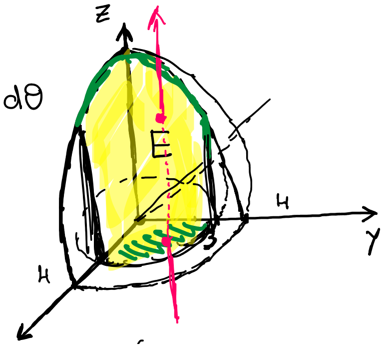
$$\frac{5}{\cancel{3}_{1}} \cdot \cancel{63}^{21} \left[\sin\theta - \cos\theta\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = 105\left[-1-1\right] = -210$$

Problem 6. Find the volume of the sold that is above the xy plane, below the ellipsoid $4x^2+4y^2+z^2=64$ but inside the cylinder $x^2+y^2=9$.

$$z = \sqrt{64-4(x^2+y^2)}$$

$$4x^2 = 64$$
$$x^2 = 64/4$$
$$x^2 = 16$$

$dz \, r \, dr \, d\theta$



$$\text{Vol}(E) = \iiint_E 1 \, dV =$$

$$= \int^{2\pi} \int^{3} \int^{2\sqrt{16-r^2}} 1 \, dz \, r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^3\int_0^{2\sqrt{16-r^2}} 1 \, dz \, r \, dr \, d\theta$$

$$\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 3 \\ 0 \le z \le \sqrt{64-4r^2} \end{cases}$$

$2\sqrt{16-r^2}$

$$2\pi \int_0^3 2\sqrt{16-r^2} \, r \, dr =$$

$u = 16 - r^2$

$du = -2r \, dr$

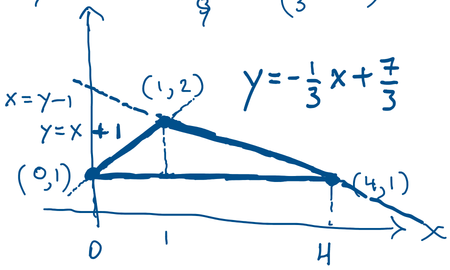
$$-2\pi \int_{16}^{7} u^{1/2} \, du =$$

$$= -2\pi \, \frac{2}{3} \, u^{3/2} \Big|_{16}^{7} = -\frac{4\pi}{3}\left(7^{3/2} - 16^{3/2}\right) =$$

$$= \frac{4\pi}{3}\left(64 - 7^{3/2}\right)$$

Problem 7. Let D be the triangular region with vertices $(0,1)$, $(1,2)$, and $(4,1)$. Set up but do not evaluate $\int\int_D 7y^2 \, dA$ in the order $dy\,dx$ and $dx\,dy$.

$\frac{4}{3}x = \left(\frac{7}{3} - y\right)(3)$



$$\int_0^1\int_1^{x+1} 7y^2 \, dy \, dx + \int_1^4\int_1^{-\frac{1}{3}x+\frac{7}{3}} 7y^2 \, dy \, dx =$$

$$= \int_1^2\int_{y-1}^{7-3y} 7y^2 \, dx \, dy$$

Problem 8. Let $D = \{(x,y) : 0 \le x \le 1, 0 \le y \le x^2\}$. Evaluate $\displaystyle\iint_D \frac{5y}{6x^5+1}\,dA$.

dy dx

$$\int_0^1 \int_0^{x^2} \frac{5y}{6x^5+1}\, dy\, dx$$

$$\frac{5}{6x^5+1}\;\frac{1}{2}\Big[y^2\Big]_{y=0}^{y=x^2} = \frac{5}{2}\,\frac{1}{6x^5+1}\,(x^4-0)$$

$$\int_0^1 \frac{5}{2}\,\frac{x^4}{6x^5+1}\, dx =$$

$u = 6x^5+1$

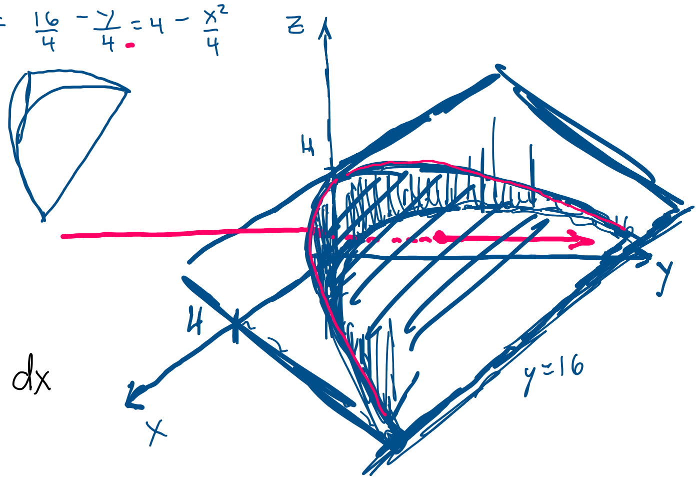
$du = 30x^4\, dx$

$$= \frac{\cancel{5}^{\,1}}{2}\int_1^7 \frac{1}{\cancel{30}_{\,6}}\,\frac{1}{u}\, du = \frac{1}{12}\; \ln u \Big|_1^7 = \frac{1}{12}\ln 7 - \ln 1 =$$

$$= \frac{1}{12}\ln 7.$$

Problem 9. Express $\displaystyle\iiint_E f(x,y,z)\, dV$ in the order $dydzdx$ if E is the solid bounded by $y = x^2$, $z = 0$, $y + 4z = 16$. $\rightsquigarrow$ $z = \frac{16}{4} - \frac{y}{4} = 4 - \frac{x^2}{4}$

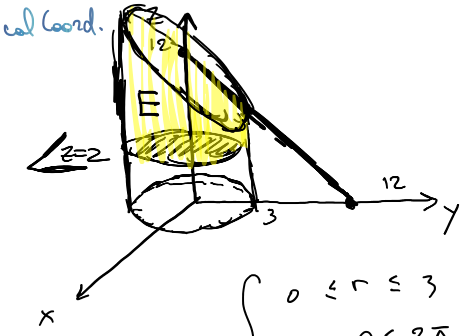
$$\begin{cases} x^2 \le y \le 16 - 4z \\ -4 \le x \le 4 \\ 0 \le z \le 4 - \frac{x^2}{4} \end{cases}$$



$$\int_{-4}^{4}\int_0^{4-\frac{x^2}{4}}\int_{x^2}^{16-4z} f(x,y,z)\, dy\, dz\, dx$$

Problem 10. Find the volume of the solid that is enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 12$ and $z = 2$.

Cylindrical Coord.



$$\mathrm{Vol}(E) = \iiint_E 1\, dV =$$

$$= \int_0^{2\pi}\int_0^{3}\int_2^{12 - r\sin\theta} 1\; dz\; r\, dr\, d\theta =$$

$$\begin{cases} 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \\ 2 \le z \le 12 - y \\ 2 \le z \le 12 - r\sin\theta \end{cases}$$

$$(12 - r\sin\theta - 2)\, r\, dr$$

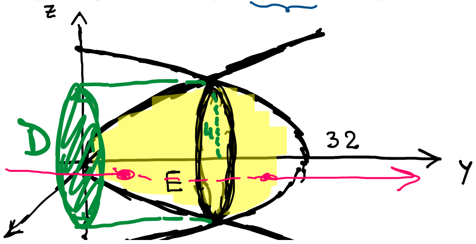
$$\int_0^3 (10r - r^2\sin\theta)\, dr = \left[5r^2 - \frac{1}{3}r^3\sin\theta\right]_{r=0}^{r=3} =$$

$$= \int 45 - \frac{27}{3}\sin\theta - 0 \quad d\theta$$

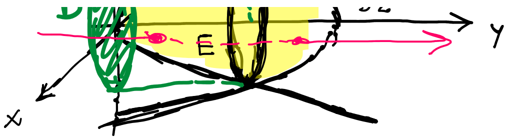
$$45\theta + \frac{27}{3}\cos\theta \Big|_0^{2\pi} = 45 \cdot 2\pi = 90\pi .$$

Problem 11. Find the volume of the solid enclosed by the paraboloids $y = \overset{r^2}{\underbrace{x^2 + z^2}}$ and $y = \underbrace{32 - x^2 \overset{r^2}{-} z^2}$.

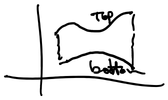
$$\mathrm{Vol}(E) = \iint_D (\text{Right} - \text{Left})\, dA =$$



D



$$= \int_0^{2\pi}\int_0^4 \left[(32-r^2)-r^2\right] r\,dr\,d\theta$$



$$\int_0^4 (32r - 2r^3)\,dr$$

$$16r^2 - \frac{2}{4}r^4 \Big|_0^4 =$$

$32 - r^2 = r^2$

$32 - 2r^2 = 0$

$16 - r^2 = 0$ $\boxed{r=4}$

$$= 16\cdot 16 - \frac{1}{\cancel{8}_1}\overset{2}{\cancel{4}}\cdot 4^3$$

64

128

$= 256 - 128 = 128$

... Times 2π

$$\boxed{256\pi}$$

$$\text{Vol}(E) = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^4\int_{r^2}^{32-r^2} 1\,dy\,r\,dr\,d\theta$$

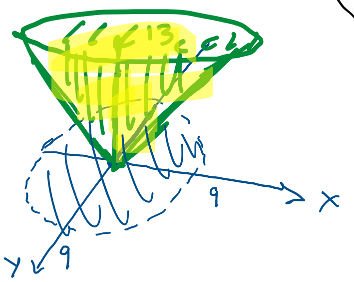
Problem 12. Convert to Cylindrical: $\int_{-9}^{9}\int_{-\sqrt{81-y^2}}^{\sqrt{81-y^2}}\int_{\sqrt{x^2+y^2}}^{13} xz\,dz\,dx\,dy.$

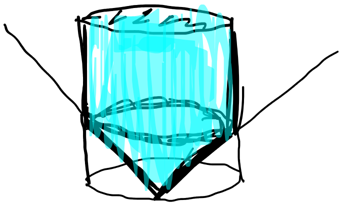
$\sqrt{x^2+y^2} \le z \le 13$

$\sqrt{x^2+y^2} = 13$

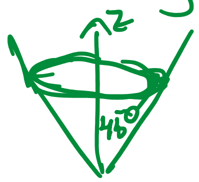
$x^2+y^2 = 13^2$

$r \le z \le 13$





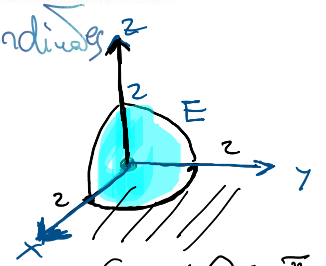
$z = r$ in Cyl. Coord



$$\int_0^{2\pi}\int_0^{9}\int_r^{13} r(\cos\theta)\,z\,dz\,r\,dr\,d\theta$$

Problem 13. Find $\iiint_E (\overset{\rho^2}{x^2+y^2+z^2})\,dV$ where E is the part of the ball centered at the origin with radius 2 in the first octant.

Spherical Coordinates



$$\int_0^{\frac{\pi}{2}}\int_0^{\pi/2}\int_0^2 (\rho^2)\,\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$\begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le \rho \le 2 \\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$

$$\left(\frac{\pi}{2}-0\right)\cdot\frac{1}{5}\Big[\rho^5\Big]_0^2\cdot\Big[-\cos\varphi\Big]_0^{\frac{\pi}{2}} =$$

$$= \frac{\pi}{\underset{1}{\cancel{2}}}\,\frac{1}{5}\left(\overset{16}{\cancel{32}}-0\right)\left(-0--1\right) = \frac{16}{5}\pi$$

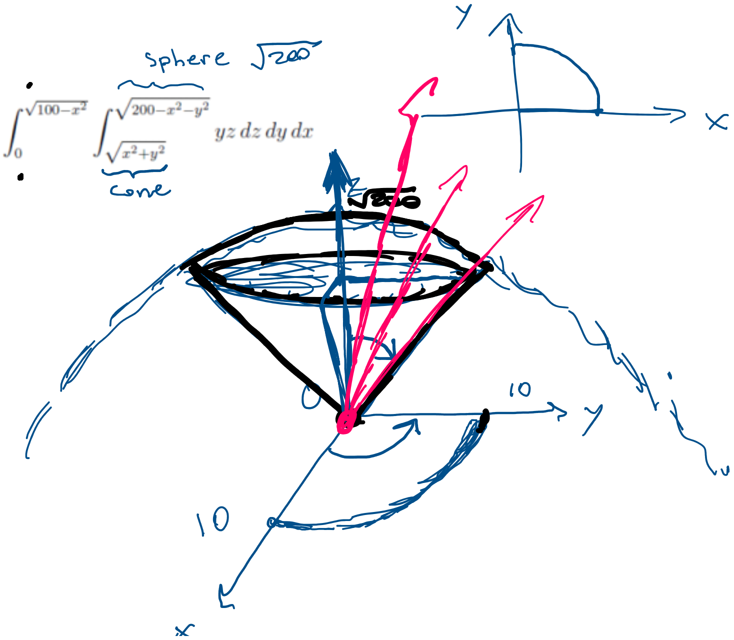
Problem 14. Evaluate in spherical coordinates. $\int_0^{10}\int_0^{\sqrt{100-x^2}}\int_{\underbrace{\sqrt{x^2+y^2}}_{\text{cone}}}^{\overbrace{\sqrt{200-x^2-y^2}}^{\text{sphere } \sqrt{200}}} yz\,dz\,dy\,dx$

$200 - r^2 = r^2$

$200 = 2r^2$

$r^2 = 100$

$$\begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le \varphi \le \frac{\pi}{4} \\ 0 \le \rho \le \sqrt{200} \end{cases}$$



$0 \le \varphi \le \quad 4$

$0 \le \rho \le \sqrt{200}$

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{200}} (\rho \sin\varphi \sin\theta)(\rho\cos\varphi)\,\rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta$$

$$\int \sin^2\varphi\cos\varphi\, d\varphi$$

$$\int u^2\, du$$

$$\left[\frac{\rho^5}{5}\right]_0^{\sqrt{200}} \left[\frac{\sin^3\varphi}{3}\right]_0^{\pi/4} \left[-\cos\theta\right]_0^{\frac{\pi}{2}} =$$

$$= \frac{1}{5}\left(200^{5/2}\right)\frac{1}{3}\left[\left(\frac{\sqrt{2}}{2}\right)^3 - 0\right]\left[-0 - -1\right]$$

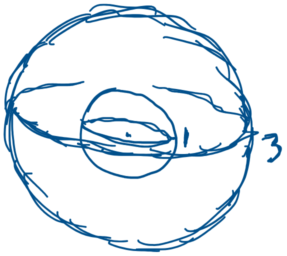
$$\sqrt{2\cdot 10^2} = 10\sqrt{2}$$

$$\frac{1}{5}\,\frac{1}{3}\,10^5 \cdot \cancel{4}\sqrt{2} \cdot \frac{\cancel{2}\sqrt{2}}{\cancel{8}} =$$

$$= \frac{2^5 \cdot 5^{\cancel{5}4}}{\cancel{5}\cdot 3} \cdot 2 = \frac{64 \cdot 5^4}{3}$$

Problem 15. Let E be the region that lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$. Set up but do not evaluate $\iiint_E (x+y+z)\,dV$ in spherical coordinates.

$$\int_0^{2\pi}\int_0^{\pi}\int_1^{3} \rho\left(\sin\varphi\cos\theta + \sin\varphi\sin\theta + \cos\varphi\right)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$



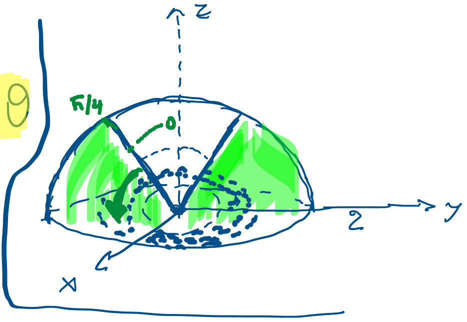
$1 \le \rho \le 3$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

Problem 16. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy plane and below the cone $z = \sqrt{x^2 + y^2}$.

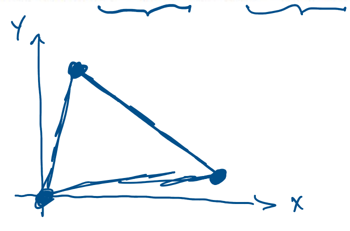
$$\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\pi/2}\int_0^2 1\,\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

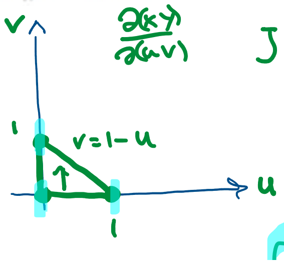


$$2\pi \quad \frac{1}{3}\left[\rho^3\right]_0^2 \left[-\cos\varphi\right]_{\frac{\pi}{4}}^{\pi/2} =$$

$$= \frac{2}{3}\pi\,(8-0)\left[0 - -\frac{\sqrt{2}}{2}\right] = \frac{\overset{8}{\cancel{16}}}{3}\pi\,\frac{\sqrt{2}}{\underset{1}{\cancel{8}}} = \frac{8}{3}\pi\sqrt{2}.$$

Problem 17. Let R be the triangular region with vertices $(0,0)$, $(9,1)$, $(1,9)$. Using the transformation $x = 9u + v$ and $y = u + 9v$ find $\int\int_R (x - 10y)dA$.

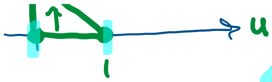




$$\frac{\partial(x,y)}{\partial(u,v)} \qquad J = \begin{vmatrix} 9 & 1 \\ 1 & 9 \end{vmatrix} = 80$$

(9u + v (−9)

$u = 9x - y$



$x = 9u + v$ (-9)

$y = u + 9v$ (-9)

$-9x = -81u - 9v$

$y - 9x = -80u$

$-9y = -9u - 81v$

$x - 9y = -80v$

$$\left\{ \begin{array}{l} u = \frac{9x}{80} - \frac{y}{80} \\ v = \frac{9y}{80} - \frac{x}{80} \end{array} \right.$$

$$\int_0^1 \int_0^{1-u} (9u + v - 10u - 90v)\, 80\, dv\, du$$

$-u - 89v$

$$= -80 \int \left(u - u^2 + \frac{89}{2} - 89u + \frac{89}{2}u^2\right) du$$

$$80 \left[-uv - \frac{89v^2}{2} \right]_{v=0}^{v=1-u}$$

$$-80 \left[u(1-u) + \frac{89}{2}(1-u)^2 \right] =$$

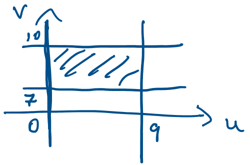
$(1 - 2u + u^2)$

$u = 0$ $u = 9$ $v = 7$

Problem 18. Let R be the parallelogram enclosed by the lines $x - 6y = 0$, $x - 6y = 9$, $6x - y = 7$, $6x - \overset{v=10}{y} = 10$. Using the transformation $u = x - 6y$ and $v = 6x - y$, find $\int\int_R 9\frac{x-6y}{6x-y} dA$

$$\int_0^9 \int_7^{10} 9 \frac{u}{v} \frac{1}{35} dv\, du$$

$$J = \begin{vmatrix} -\frac{1}{35} & \frac{6}{35} \\ -\frac{6}{35} & \frac{1}{35} \end{vmatrix} = -\frac{1}{35^2} + \frac{36}{35^2} = \frac{1}{35}$$



$$\frac{9}{35} \left[\ln v \right]_7^{10} \frac{1}{2} \left[u^2 \right]_0^9 =$$

$$= \frac{9}{70} (\ln 10 - \ln 7)(81 - 0)$$

$u = x - 6y$ (-6)

$v = 6x - y$ (-6)

$-6u = -6x + 36y$

$-6v = -36x + 6y$

$u - 6v = -35x$

$v - 6u = 35y$

$$x = \frac{6v}{35} - \frac{u}{35}$$

$$y = \frac{v}{35} - \frac{6u}{35}$$

Problem 19. Let R be the region bounded by $25x^2 + 4y^2 = 100$. Using the transformation $x = 2u$ and $y = 5v$, find $\underbrace{\int\int_R 4x^2 dA}$.

$25 \cdot 4u^2 + 4 \cdot 25v^2 = 100$

$$\boxed{u^2 + v^2 = 1}$$

$\iint$

$$\boxed{u^2+v^2=1}$$

$$\iint 4(4u^2)\cdot 10\;du\,dv$$

Unit disk
in uv-plane

$$J=\begin{vmatrix}2 & 0\\ 0 & 5\end{vmatrix}=10$$

$u=r\cos\theta$

Polar

$$\frac{1}{2}(1+\cos 2\theta)$$

$$\int_0^{2\pi}\int_0^1 16(r^2\cos^2\theta)\cdot 10\,r\,dr\,d\theta =$$

$$\overset{20}{\cancel{\overset{40}{\cancel{160}}}}\;\frac{1}{\cancel{4}}\Big[r^4\Big]_0^1\;\frac{1}{\cancel{2}}\Big[\theta+\frac{1}{2}\sin 2\theta\Big]_0^{2\pi} =$$

$$= 20\,(1-0)\,(2\pi\pm 0) = 40\pi.$$