Wir 6: Exam 2 Review

Sections 14.1, 14.3-14.8

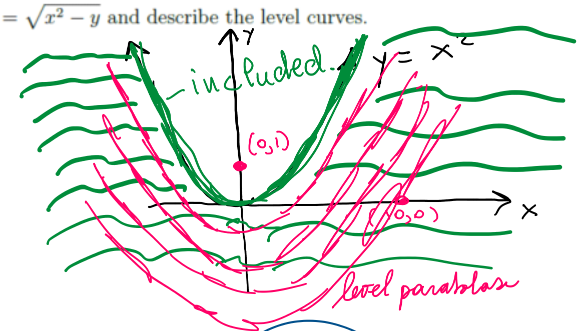
Problem 1. Sketch the domain of $f(x,y)=\sqrt{x^2-y}$ and describe the level curves.

$x^2-y \geq 0$

$x^2 \geq y$

$0^2 \not\geq 1$ NO

$10^2 \geq 0$ ✓



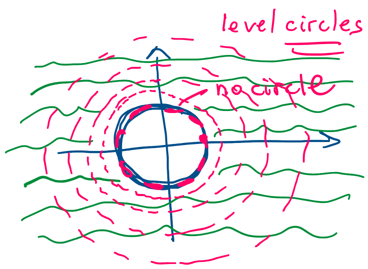
$\sqrt{x^2-y} = k \geq 0 \Rightarrow x^2 - y = k^2$

$y = x^2 - k^2$

Problem 2. Sketch the domain of $f(x,y)=\ln(y^2+x^2-1)$ and describe the level curves.

$y^2+x^2-1 > 0$

$y^2+x^2 > 1$



$e^{\ln(y^2+x^2-1)} = e^k$

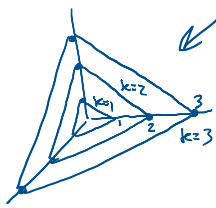
$y^2+x^2-1 = e^k \Rightarrow x^2+y^2 = 1+e^k$

radius is always >1

Problem 3. What are the level surfaces to the equation $f(x, y, x) = x + y + z$? PLANES

$f(x, y, z) = k$

$x + y + z = k$ it is a plane



Problem 4. $f(x, y) = \sin(x^2 + y^2)$, find all first and second partial derivatives.

$f_x = [2x]\cos(x^2+y^2)$

$f_y = [2y]\cos(x^2+y^2)$

$f_{xy} = -2x\,2y\sin(x^2+y^2)$

$f_{yx} = -2y\,2x\sin(x^2+y^2)$

$f_{xx} = 2\cos(x^2+y^2) - 2x\,2x\sin(x^2+y^2)$

$f_{yy} = 2\cos(x^2+y^2) - 2y\,2y\sin(x^2+y^2)$

$f(x,y)$

Problem 5. Find an equation for the tangent plane to the surface $z = 2x^2 + y^2$ at the point $(1, 1)$

$$z - \overset{f(P_0)}{z_0} = f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0)$$

$f(1,1) = 2 + 1 = 3$

$f_x = 4x \quad 4$

$f_y = 2y \quad 2$

$$\boxed{z - 3 = 4(x-1) + 2(y-1)}$$

simplify...

Problem 6. Find the tangent plane (equation) to the surface $\underbrace{2xy + 3yz + 7xz = -9}_{F(x,y,z) = -9}$ at the point $\overbrace{(1, 2, -1)}^{P_0}$.

$4 - 6 - 7 = -9$ ✓

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$$

$$\left.\begin{aligned} F_x &= 2y + 7z \\ F_y &= 2x + 3z \\ F_z &= 3y + 7x \end{aligned}\right\} \text{at } P_0 \quad \begin{aligned} &= 4 - 7 = -3 \\ &2 - 3 = -1 \\ &6 + 7 = 13 \end{aligned}$$

$$-3(x-1) - 1(y-2) + 13(z+1) = 0$$

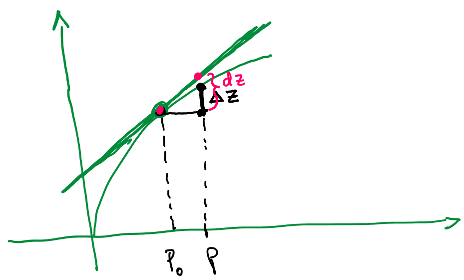
simplify...

Problem 7. If $z = x^3y^2$, find the differential, dz, and explain what it measures.

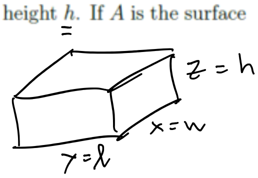
$$dz = z_x\, dx + z_y\, dy =$$

$$= 3x^2y^2\, dx + 2x^3y\, dy$$

$$dz \approx \Delta z$$



Problem 8. Consider a rectangular box with length l, width w and height h. If A is the surface area of the box, find the differential, dA.



$$A = 2xy + 2xz + 2yz$$

$$dA = A_x\, dx + A_y\, dy + A_z\, dz$$

$$(2y+2z)dx + (2x+2z)dy + (2x+2y)dz = dA$$

Problem 9. The height of a cone was measured to be 3 cm with a maximum error of 0.1 cm and the radius of the cone is measured to be 2 cm with a maximum error of 0.2 cm. Use differentials to estimate the maximum error in the calculated volume of the cone.

$$V = \frac{1}{3}\pi r^2 h$$

$h = 3\text{ cm}$ $\quad dh = \frac{1}{10}\text{ cm}$

$r = 2\text{ cm}$ $\quad dr = \frac{2}{10}\text{ cm}$

$$dV = V_r\,dr + V_h\,dh$$

$$\left(\frac{2}{3}\pi r h\right)\frac{2}{10} + \left(\frac{\pi}{3} r^2\right)\frac{1}{10}$$

$$\frac{2}{3}\pi\, 6\, \frac{2}{10} + \frac{\pi}{3}\cdot 4\cdot\frac{1}{10} = \frac{24\pi + 4\pi}{30} =$$

$$= \frac{\overset{14}{\cancel{28}}}{\underset{15}{\cancel{30}}}\pi\ \text{cm}^3 = \frac{14}{15}\pi\ \text{cm}^3$$

Problem 10. Use a linear approximation (tangent plane) to estimate $((\overset{x}{2.1})^2 + (\overset{y}{0.1})^3)^3$

$$f(x,y) = (x^2+y^3)^3$$

$f_x = 3(\quad)^2\cdot 2x$ $\quad f_y = 3(\quad)^2\, 3y^2$ $\quad P_0(\overset{x_0}{2}, \overset{y_0}{0})$

$$f(P_0) = (4+0)^3 = 4^3 = 2^6 = 64 = z_0$$

$$z = z_0 + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0)$$

$$z = L(x,y) = 64 + \underbrace{6x(x^2+y^3)^2}_{\text{at } P_0}(2.1-2) + \boxed{\underbrace{9y^2(x^2+y^3)^2}_{\text{at } P_0}\underbrace{(0.1-0)}_{\text{ZERO}}} =$$

$$= 64 + 12\cdot\overset{8}{\cancel{16}}\,\frac{1}{\underset{5}{\cancel{10}}} + \cancel{\emptyset} =$$

$$= 64 + \underline{96} = \underline{320+96} = \underline{416}$$

CLOSE!

$$= 64 + \frac{96}{5} = \frac{320+96}{5} = \frac{416}{5}$$ ← CLOSE!

Problem 11. Use differentials to approximate $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$ ← 6.991523

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$ $P_0(3, 2, 6)$ with x_0, y_0, z_0

$\sqrt{9+4+36} = \sqrt{49} = 7$

$f(P_0) = 7$

$df = f_x\,dx + f_y\,dy + f_z\,dz$

$$\frac{\cancel{2}x_0}{\cancel{2}\sqrt{x_0^2+y_0^2+z_0^2}}(x-3) + \frac{y_0}{\sqrt{x_0^2+y_0^2+z_0^2}}(y-2) + \frac{z_0}{\sqrt{x_0^2+y_0^2+z_0^2}}(z-6)$$

($x = 3.02$, $y = 1.97$, $z = 5.99$)

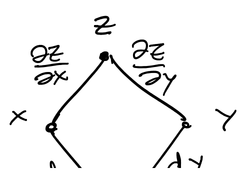
$$\frac{3}{7}\,\frac{2}{100} + \frac{2}{7}\left(-\frac{3}{100}\right) + \frac{6}{7}\left(-\frac{1}{100}\right)$$

$$\frac{\cancel{6} - \cancel{6} - 6}{700}$$ error $+ 7$

Answer: $7 - \frac{6}{700} = \frac{4900-6}{700} = \boxed{\frac{4894}{700}} = 6.99$ ← very close!

Problem 12. If $z = e^{x^2+y^2}$, $x = e^t$. $y = \cos t$, find $\frac{dz}{dt}$

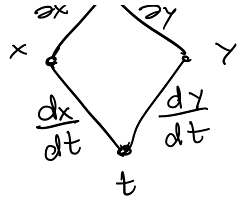
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} =$$



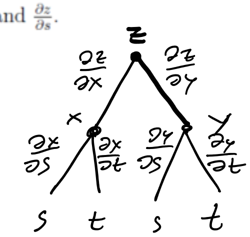
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} =$$

$$= 2x e^{x^2+y^2} e^t - 2y e^{x^2+y^2} \sin t$$

minus



Problem 13. For $z = xy$, $x = \cos(st^2)$, $y = \sin e^t$, find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$.



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} =$$

$$= y\, 2st(-\sin st^2) + x e^t \cos e^t$$

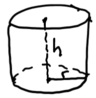
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \underbrace{y\, t^2(-\sin st^2)} + x \cdot \emptyset$$

Problem 14. The radius and height of a circular cylinder change with time. When the height is 1 meter and the radius is 2 meters, the radius is increasing at a rate of 4 meters per second and the height is decreasing at a rate of 2 meters per second. At that same instant, find the rate at which the volume is changing

when $h = 1m$, $r = 2m$

$$\left.\begin{array}{l}\frac{dh}{dt} = -2\,\frac{m}{sec} \\ \frac{dr}{dt} = 4\,\frac{m}{sec}\end{array}\right\} \quad \frac{dV}{dt} = ?$$



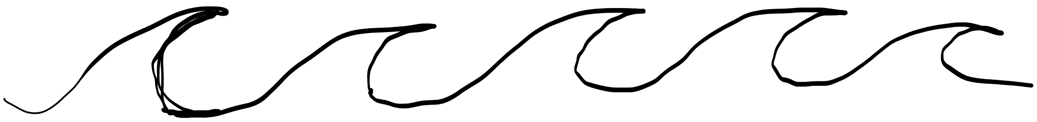
$V = \pi r^2 h$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$2\pi r h \cdot 4 + \pi r^2 (-2)$$

$$2\pi \cdot 2 \cdot 4 - \pi 4 \cdot 2 =$$

$$= 16\pi - 8\pi = 8\pi \,\frac{m^3}{sec}$$



Problem 15. Let $f(x, y) = \sqrt{xy}$. Find the directional derivative of f at the point $P(4, 1)$ in the direction from P to $Q(6, 2)$

$\vec{v} = \vec{PQ} = Q - P =$

$= \langle 2, 1 \rangle$ $\quad |\langle 2,1\rangle| = \sqrt{5}$

$\vec{u} = \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$

$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u}$

$\vec{\nabla} f = \langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \rangle$

at $(4,1) = \langle \frac{1}{4}, 1 \rangle$

Answer: $\langle \frac{1}{4}, 1 \rangle \cdot \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle =$

$$= \frac{2}{4\sqrt{5}} + \frac{1}{\sqrt{5}} = \frac{2+4}{4\sqrt{5}} = \frac{3}{2\sqrt{5}} \text{ OR } \frac{3\sqrt{5}}{10}.$$

Problem 16. Let $f(x, y) = \sqrt{xy}$. What is the direction of the largest rate of change at the point $P(4, 1)$?

Answer: It is the direction of $\vec{\nabla} f(4,1)$.

$\vec{\nabla} f = < \frac{y}{2\sqrt{xy}} , \frac{x}{2\sqrt{xy}} >$ at $(4, 1)$

$\sqrt{4\cdot 1} = 2$

$< \frac{1}{2\cdot 2} , \frac{4}{2\cdot 2} > = \underbrace{< \frac{1}{4}, 1 >}_{\text{Answer:}}$

direction of $< \frac{1}{4}, 1 >$.

Problem 17. Let $f(x, y) = e^{x+y}$. What is the maximum rate of change at the point $P(-1, 1)$?

Answer: $| \vec{\nabla} f(-1,1) |$

$\vec{\nabla} f = < e^{x+y} , e^{x+y} >$ at $(-1,1)$ it is $<1, 1>$

$|<1,1>| = \sqrt{2}$ = Answer

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Problem 18. For the $f(x, y) = 2x^3 - xy^2 + 5x^2 + y^2 + 5$, find all local minima, maxima, and saddle points.

$\boxed{\text{Step 1}}$ $\vec{\nabla} f = \vec{0}$ so $\langle 6x^2 - y^2 + 10x, -2xy + 2y \rangle = \langle 0, 0 \rangle$

$$\begin{cases} 6x^2 - y^2 + 10x = 0 \\ 2y(-x+1) = 0 \Rightarrow \text{either } y = 0 \text{ or } x = 1 \end{cases}$$

If $\boxed{y = 0} \Rightarrow 6x^2 + 10x = 0 \Rightarrow 2x(3x+5) = 0 \Rightarrow x = 0$ OR $x = -\frac{5}{3}$

$(0,0)$ $(-\frac{5}{3}, 0)$ $(1,-4)$ $(1,4)$

If $\boxed{x = 1} \Rightarrow 6 - y^2 + 10 = 0 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$

$\boxed{\text{Step 2}}$ $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x+10 & -2y \\ -2y & -2x+2 \end{vmatrix}$ $\quad \frac{10}{3} + \frac{6}{3}$ $\quad 4 \cdot 12 (-\frac{5}{3})$

$D(0,0) = \begin{vmatrix} 10 & 0 \\ 0 & 2 \end{vmatrix} = 20 > 0$ $f_{xx} = 10 > 0$ ($f'' > 0$)

$D(-\frac{5}{3}, 0) = \begin{vmatrix} -10 & 0 \\ 0 & \frac{16}{3} \end{vmatrix} < 0$

$D(1,-4) = D(1,4) = \begin{vmatrix} 22 & \pm 8 \\ \pm 8 & 0 \end{vmatrix} = -64 < 0$

$(0,0)$ Local Min
$(-\frac{5}{3}, 0)$ Saddle Pt.
$(1,-4)$ and $(1,4)$ are Saddle Pts

GRAPH with Desmos or Geogebra, etc...

Problem 19. Find the absolute maximum and minimum values of $f(x, y) = 7 + xy - x - 2y$ over the closed triangular region with verticies $(1, 0)$, $(5, 0)$, $(1, 4)$.

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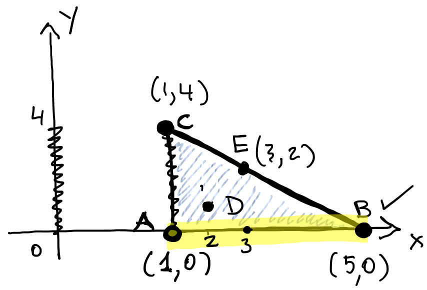


Step 1: $\vec{\nabla} f = \vec{0}$ (INSIDE)

$\vec{\nabla} f = \langle y-1,\ x-2 \rangle = \langle 0,0 \rangle$ at $(2,1) = D$

Step 2: $\overline{AB}$ $\overline{BC}$ $\overline{CA}$

$\overline{AB}$: $y = 0 \Rightarrow f(x) = 7 - x$

$f'(x) = -1$ $\quad 1 \le x \le 5$

$x = 1$ $\quad x = 5$ $\quad (1,0)$ $\quad (5,0)$

$\overline{BC}$: $\boxed{y = -x + 5}$ $\quad 1 \le x \le 5$

$f(x) = 7 + x(5-x) - x - 2(5-x) =$

$= 7 + 5x - x^2 - x - 10 + 2x =$

$= -x^2 + 6x - 3$

$f'(x) = -2x + 6 = 0$ when $x = 3$

$y = -3 + 5 = 2$ $\quad (3,2)$ E

$\overline{CA}$: $x = 1$ $\quad f(y) = 7 + y - 1 - 2y = 6 - y$

$0 \le y \le 4$ $\quad f'(y) = -1$

$(1,0)$ $\quad (1,4)$

$f(2,1) = 7 + \cancel{2} - \cancel{2} - 2 = 5$

$f(1,0) = 7 - 1 = 6$ ✓

$f(5,0) = 7 - 5 = 2$

$f(3,2) = -9 + 18 - 3 = 6$

$f(1,4) = -1 + 6 - 3 = 2$

$f(1,0) = 6$ ✓

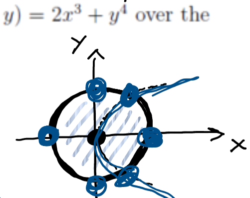
$f(1,4) = 6 - 4 = 2$ ✓

Step 3 Absolute Max is 6 at $(1,0)$ & $(3,2)$.

Absolute Min is 2 at $(5,0)$ & $(1,4)$.

g: $x^2 + y^2 = 1$ constraint

Problem 20. Find the absolute maximum and minimum values of $\underbrace{f(x,y) = 2x^3 + y^4}_{f}$ over the region $D = \{(x,y) : x^2 + y^2 \le 1\}$



Inside: $\vec{\nabla} f = \vec{0}$ $\quad \langle 6x^2, 4y^3 \rangle = \langle 0, 0 \rangle$ at $(0,0)$

$\boxed{f(0,0) = 0}$

Boundary Circumference (Lagrange Mult.)

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$\langle 6x^2, 4y^3 \rangle = \lambda \langle 2x, 2y \rangle$

$\begin{cases} 3x^2 = \lambda x \\ 2y^3 = \lambda y \\ x^2 + y^2 = 1 \end{cases}$ $\quad \begin{cases} x(3x - \lambda) = 0 \\ y(2y^2 - \lambda) = 0 \end{cases}$

If $x \ne 0$ $\quad y \ne 0$ $\quad \lambda = 3x$, $\lambda = 2y^2$ $\quad 3x = 2y^2$ $\quad \boxed{x = \frac{2}{3} y^2}$

Constraint $\quad \frac{4}{9} y^4 + y^2 = 1 \Rightarrow$ $\quad 4y^4 + 9y^2 = 9$

$z = y^2$ $\quad 4z^2 + 9z - 9 = 0$ $\quad$ so $z = \frac{-9 \pm \sqrt{81 + 144}}{8} =$

$\cancel{-\frac{24}{8} = -3}$

$\sqrt{225}$ $\quad -9+15$

15
25
75
15
225

36
4
144
81
225

$z = y^2$

$4z^2 + 9z - 9 = 0$ so $z = \frac{-9 \pm \sqrt{81+144}}{8} = \frac{-9 \pm \sqrt{225}}{8} = \frac{-9 \pm 15}{8} = \cancel{-\frac{24}{8} = -3}$ or $\frac{6}{8} = \frac{3}{4}$

$z = y^2 = \frac{3}{4}$ $y = \pm \frac{\sqrt{3}}{2}$

$x = \frac{2}{3}y^2 = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ $f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = f\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \frac{2}{8} + \frac{9}{16} = \frac{4+9}{16} = \frac{13}{16}$

$(0,1)$ $(0,-1)$ $(1,0)$ $(-1,0)$ $f(0,\pm 1) = 1$, $\underbrace{f(+1,0) = 2}_{\text{ABSOLUTE MAX}}$, $\underbrace{f(-1,0) = -2}_{\text{absolute min}}$

Problem 21. Use the method of Lagrange to find the maximum and minimum values of $f(x, y) = 6x + 6y$ subject to the constraint $x^2 + y^2 = 18$. g

$\vec{\nabla} f = \lambda \vec{\nabla} g$ $\langle 6, 6 \rangle = \lambda \langle 2x, 2y \rangle$

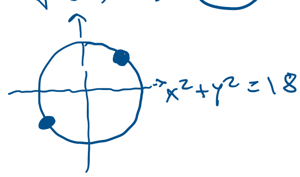
$\begin{cases} 3 = \lambda x \\ 3 = \lambda y \\ x^2 + y^2 = 18 \end{cases}$ $3 = \boxed{\lambda x = \lambda y} \Rightarrow \lambda (x - y) = 0 \Rightarrow$ either $\cancel{\lambda = 0}$ OR $x = y$

If $\boxed{x = y} \xrightarrow{\text{constraint}}$ $x^2 + x^2 = 18 \Rightarrow 2x^2 = 18 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

$(3,3)$ $(-3,-3)$

$f(3,3) = 18 + 18 = 36$ Absolute Max value

$f(-3,-3) = -36$ Absolute min value



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Problem 22. Use the method of Lagrange to find the maximum and minimum values of $f(x, y) = y^2 - x^2$ subject to the constraint $\frac{1}{4}x^2 + y^2 = 25$ g

$\vec{\nabla} f = \lambda \vec{\nabla} g$ $\langle -2x, 2y \rangle = \lambda \langle \frac{1}{2}x, 2y \rangle$

$-2x = \frac{1}{2}\lambda x \Rightarrow x\left(-2 - \frac{1}{2}\lambda\right) = 0$ either $x = 0$ OR $\lambda = -4$

$y(1+4) = 0 \Rightarrow y = 0$

$2y = 2\lambda y \Rightarrow y(1 - \lambda) = 0$ either $y = 0$ OR $\lambda = 1$

$\frac{1}{4}x^2 + y^2 = 25$

$\cancel{2}y = \cancel{2}\lambda y \Rightarrow y(1-\lambda)=0$ either $y=0$ OR $\lambda=1$

$\frac{1}{4}x^2+y^2=25$

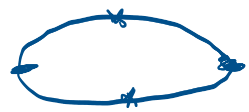
If $y=0 \Rightarrow \frac{1}{4}x^2+0^2=25 \Rightarrow x^2=100 \Rightarrow x=\pm 10$ $(10,0)$ $(-10,0)$

If $\lambda=1 \Rightarrow -2x=\frac{1}{2}x \Rightarrow \underline{x=0} \Rightarrow \frac{1}{4}0^2+y^2=25 \Rightarrow y=\pm 5$

$(0,5)$ $(0,-5)$

$f(\pm 10, 0) = -100$ absolute min

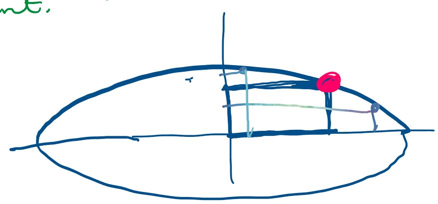
$f(0,\pm 5) = 25$ absolute max



Problem 23. Find the volume of the largest rectangular box with faces parallel to the coordinate planes than can be inscribed in the ellipsoid $\underbrace{16x^2+4y^2+9z^2=144.}_{\text{constraint.}}$ g

Maximize volume

$V = xyz$



$\vec{\nabla} V = \lambda \vec{\nabla} g$

$\langle yz, xz, xy\rangle = \lambda \langle 32x, 8y, 18z\rangle$

$\begin{cases} yz = 32\lambda x \\ xz = 8\lambda y \\ xy = 18\lambda z \\ \text{constraint} \end{cases}$

$\lambda = \frac{yz}{32x} = \frac{xz}{8y} = \frac{xy}{18z}$

$\cancel{8}y^2z = \overset{4}{\cancel{32}}x^2z \Rightarrow y^2 = 4x^2$

$\overset{9}{\cancel{18}}xz^2 = \overset{4}{\cancel{8}}xy^2 \Rightarrow 9z^2 = 4y^2 \Rightarrow y^2 = \frac{9}{4}z^2$

$x^2 = \frac{y^2}{4}$ $z^2 = \frac{4}{9}y^2$

$\overset{4}{\cancel{16}}\frac{y^2}{\cancel{4}} + 4y^2 + \cancel{9}\frac{4}{\cancel{9}}y^2 = 144$

$12y^2 = 144 \Rightarrow y^2 = 12$

$x^2 = \frac{12}{4} = 3$ $z^2 = \frac{4}{\cancel{9}_3}\cdot \overset{4}{\cancel{12}} = \frac{16}{3}$

$y = +\sqrt{12} = \underline{2\sqrt{3}}$ $x = \sqrt{3}$ $z = \sqrt{\frac{16}{3}} = \frac{4}{\sqrt{3}}$

$V = xyz = 2\sqrt{3}\cdot\cancel{\sqrt{3}}\cdot\frac{4}{\cancel{\sqrt{3}}} = 8\sqrt{3}$

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