WIR 3 solved

Tuesday, February 9, 2021 6:56 PM

Problem 1. What is the equation of the sphere centered at $\overset{C}{(6, 4, 12)}$ with radius 6? Describe the intersection of this sphere with the three coordinate planes.

$$(x-6)^2 + (y-4)^2 + (z-12)^2 = 6^2 = 36$$

$$x^2 + y^2 + z^2 - 12x - 8y - 24z = 36 - 196 = 160$$

$$\begin{array}{r} 36 + \\ 16 \\ 144 \\ \hline 196 \end{array}$$

xy-plane: $z = 0$ $\quad (x-6)^2 + (y-4)^2 + \underbrace{(0-12)^2}_{144} = 36$

circle $C(6,4,0)$ $\quad (\quad)^2 + (\quad)^2 = \text{negative}$

radius

No intersection.

yz-plane: $x = 0$ $\quad \underbrace{(0-6)^2}_{} + (y-4)^2 + (z-12)^2 = 36$

$$(\quad)^2 + (\quad)^2 = 36 - 36 = 0$$

Just one point: $(0, 4, 12)$

xz-plane: $y = 0$ $\quad (x-6)^2 + (0-4)^2 + (z-12)^2 = 36$

$$(\quad)^2 + \quad + (\quad)^2 = 36 - 16 = 20$$

circle with $C(6, 0, 12)$ and radius $\sqrt{20}$.

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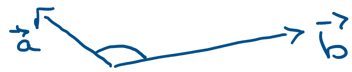


Problem 2. Let $\mathbf{a} = \langle 1, 2, -1\rangle$ and $\mathbf{b} = \langle 2, -1, 2\rangle$. Find the vector projection of $\mathbf{b}$ onto $\mathbf{a}$, that is $\text{proj}_{\mathbf{a}}\mathbf{b}$.

$$\text{comp}_{\vec a}\vec b = \frac{\vec a \cdot \vec b}{|\vec a|} = \frac{\cancel{2} - \cancel{2} - 2}{\sqrt{1+4+1}} = \frac{-2}{\sqrt{6}}$$

$$\text{proj}_a \vec b = \left(\frac{\vec a \cdot \vec b}{|\vec a|^2}\right)\vec a = \frac{-\cancel{2}^{\,1}}{\cancel{6}\,3}\langle 1, 2, -1\rangle =$$

$$= \left\langle -\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle .$$

Problem 3. Let $\mathbf{a} = \langle -2, 2, 1\rangle$. Find a vector $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ so that the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ equals -4, that is $\text{comp}_{\mathbf{a}}\mathbf{b} = -4$.

$$\text{comp}_{\vec a}\vec b = \frac{\vec a \cdot \vec b}{|\vec a|} = -4 \qquad |\vec a| = \sqrt{4+4+1} = 3$$

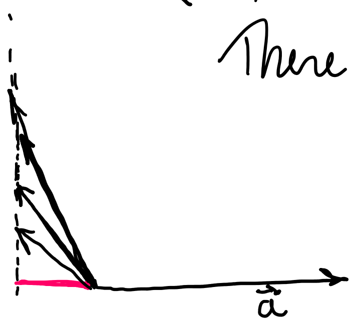
$$\frac{-2b_1 + 2b_2 + b_3}{3} = -4 \quad \text{so} \quad -2b_1 + 2b_2 + b_3 = -12$$

$\frac{-2b_1 + 2b_2 + b_3}{3} = -4$ so $-2b_1 + 2b_2 + b_3$

set $b_2 = b_3 = 0$. Then $b_1 = 6$

$\langle 6, 0, 0 \rangle$.

There are many others!



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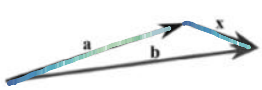
Problem 4. Use the figure below to answer the questions that follow.



$\vec{a} + \vec{x} = \vec{b}$

a.) Write $\vec{x}$ in terms of $\vec{a}$ and $\vec{b}$.

b.) If the angle between **a** and **b** is 60°, $|\mathbf{a}| = 7$, and $|\mathbf{b}| = 6$, find $\mathbf{a} \cdot \mathbf{b}$.

c.) If the angle between **a** and **b** is 60°, $|\mathbf{a}| = 7$, and $|\mathbf{b}| = 6$, find $|\mathbf{a} \times \mathbf{b}|$ and determine whether $\mathbf{a} \times \mathbf{b}$ is directed into or out of the page.

a) $\vec{x} = \vec{b} - \vec{a}$

b) $\vec{a} \cdot \vec{b} = (7)(6)\cos\frac{\pi}{3} = 7 \cdot \overset{3}{\cancel{6}} \cdot \frac{1}{\cancel{2}_1} = 21$

c) $|\vec{a} \times \vec{b}| = (7)(6)\sin\frac{\pi}{3} = 7 \cdot \overset{3}{\cancel{6}} \cdot \frac{\sqrt{3}}{\cancel{2}_1} = 21\sqrt{3}$

$\vec{a} \times \vec{b}$ points into the page.

$\vec{a} \times \vec{b}$ points into the page.

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Problem 5. Find a vector equation, a set of parametric equations, and symmetric equations for the line passing through the point $\underbrace{(-2, 3, 4)}_{P_0}$ that is parallel to the vector $\langle 1, -4, 4 \rangle = \vec{v}$

parametric equs:

$$\begin{cases} x = -2 + t \\ y = 3 - 4t \\ z = 4 + 4t \end{cases}$$

vector equ: $\langle x, y, z \rangle = \langle -2 + t, 3 - 4t, 4 + 4t \rangle$

symmetric equs:

$$t = x + 2 = \frac{y - 3}{-4} = \frac{z - 4}{4}$$

$$x + 2 = \frac{3 - y}{4} = \frac{z - 4}{4}$$

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What if it was $\perp \langle 1, -4, 4 \rangle$? $\langle a_1, a_2, a_3 \rangle$

$\boxed{a_1 - 4a_2 + 4a_3 = 0}$

What if it was $\perp \langle 1, -4, 4 \rangle$?

$\langle a_1 \; a_2 \; a_3 \rangle$

$\boxed{a_1 - 4a_2 + 4a_3 = 0}$

$\langle -4, 0, 1 \rangle \quad \langle 0, 1, 1 \rangle$

$$\begin{cases} x = -2 - 4t \\ y = 3 \\ z = 4 + t \end{cases} \quad \text{OR} \quad \begin{cases} x = -2 \\ y = 3 + t \\ z = 4 + t \end{cases}$$

Problem 6. Consider the line that passes through the points $\overset{A}{(4, 3, -1)}$ and $\overset{B}{(5, 3, 5)}$. Where does this line intersect the three coordinate planes, and if it does not intersect one of the three coordinate planes, explain why not.

$P_0 \; (5, 3, 5)$ $\quad \vec{v} = \vec{BA} = A - B = \langle -1, 0, -6 \rangle$

param eqns $\begin{cases} x = 5 - t \\ y = 3 \\ z = 5 - 6t \end{cases}$

no intersection with the xz-plane, because on line, y never equals zero (always 3)

xy-plane : $z = 0$ so $t = \frac{5}{6}$ $\quad x = 5 - \frac{5}{6} = \frac{25}{6}$

$\left(\frac{25}{6}, 3, 0\right)$

yz-plane : $x = 0$ so $t = 5$ $\quad z = 5 - 30 = -25$

$(0, 3, -25)$

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$P_0 \; (x_0, y_0, z_0)$

$\vec{n} = \langle a, b, c \rangle$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Problem 7. Find the equation of the plane that contains the point $(1, 2, -5)$ and is perpendicular

$P_0(x_0, y_0, z_0)$ $\quad a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$\vec{n}=\langle a, b, c\rangle$

Problem 7. Find the equation of the plane that contains the point $\underbrace{(1, 2, -5)}_{P_0}$ and is perpendicular to the vector $\langle -6, 4, -2\rangle = \vec{n}$

$$-6(x-1)+4(y-2)-2(z+5)=0$$

$$-6x+4y-2z-12=0$$

$+6-8-10$

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Problem 8. Find parametric equations for the line that passes through $\overbrace{(2, -1, 5)}^{P_0}$ and is

a.) parallel to the line $\dfrac{x+1}{3} = \dfrac{y-6}{4} = z$.

b.) perpendicular to the plane $8x - 11y = 2z + 6$.

$\}$ need $\vec{v}$

a) $\dfrac{x+1}{3}=t \rightarrow x = 3t-1 \qquad \dfrac{y-6}{4}=t \rightarrow y=6+4t$

$z=t \qquad$ so $\vec{v}=\langle 3, 4, 1\rangle$

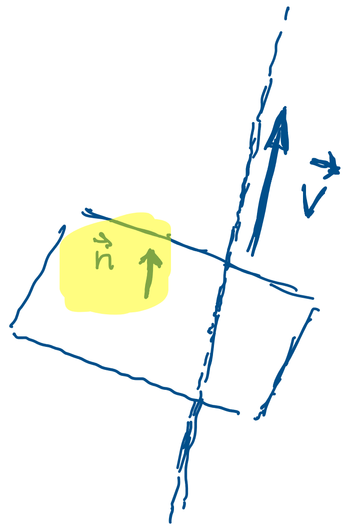
$$\begin{cases} x = 2+3t \\ y = -1+4t \\ z = 5+t \end{cases}$$

$$\begin{cases} y = -1 + 4t \\ z = 5 + t \end{cases}$$

b) $8x - 11y - 2z - 6 = 0$

$\vec{v}$ of line $= \langle 8, -11, -2 \rangle = \vec{n}$ to plane

$$\begin{cases} x = 2 + 8t \\ y = -1 - 11t \\ z = 5 - 2t \end{cases}$$



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Problem 9. Consider the triangle with vertices $P(1,0,1)$, $Q(2,3,4)$ and $R(2,1,1)$.

a.) Find the angle at the vertex Q.

b.) Find the equation of the plane that passes through the points

a) $\theta = \cos^{-1}\left(\dfrac{\vec{QP} \cdot \vec{QR}}{|\vec{QP}|\,|\vec{QR}|}\right)$ $\quad \vec{QP} = \langle -1, -3, -3 \rangle$, $\vec{QR} = \langle 0, -2, -3 \rangle$

$$\theta = \cos^{-1} \frac{0+6+9}{\sqrt{1+9+9}\,\sqrt{0+4+9}} = \cos^{-1}\left(\frac{15}{\sqrt{19}\sqrt{13}}\right)$$

$3 - 9 + 6 = 0$
$-6 + 6 = 0$

b) $P_0(1,0,1)$

$$\vec{n} = \vec{QP} \times \vec{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 3 \\ 0 & 2 & 3 \end{vmatrix} = \langle 3, -3, 2 \rangle = \vec{n}$$

$$3(x-1) - 3(y-0) + 2(z-1) = 0$$

$3(x-1)-3(y-0)+2(z\ \ )$

$3x-3y+2z-5=0$ $\quad -3\ -2$

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Problem 10. Find the equation of the plane that passes through the point $\overbrace{(1,0,1)}^{P_0}$ and

a.) is perpendicular to the line $x=9-t$, $y=7+2t$, $z=t$. $\vec{n}=\langle -1,2,1\rangle$

b.) contains line $x=9-t$, $y=7+2t$, $z=t$.

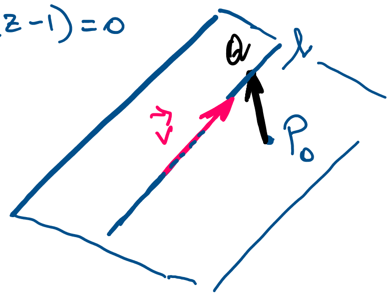
a) $-1(x-1)+2(y-0)+1(z-1)=0$

$-x+2y+z=0$

b) $P_0(1,0,1)$

$\vec{n}=\vec{v}\times\vec{P_0Q}$ $\quad \vec{v}=\langle -1,2,1\rangle$

let $Q(9,7,0)$



$\vec{P_0Q}=\langle 8,7,-1\rangle$

$$\vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & 1\\ 8 & 7 & -1\end{vmatrix}=\langle -9,+7,-23\rangle=\vec{n}$$

$-9(x-1)+7(y-0)-23(z-1)=0$

$-9x+7y-23z+32=0$ $\quad 9+23$

$$-\begin{vmatrix}-9 & 1\\ 8 & -1\end{vmatrix}=-(1-8)=$$

$=--7$

Problem 11. Consider the plane P_1 given by the equation $2x-y+3z=7$ and the plane P_2 given by the equation $3x+y+2z=3$.

$\vec{n}_1=\langle 2,-1,3\rangle$

$\vec{n}_2=\langle 3,1,2\rangle$

a.) Find the angle between the planes. ACUTE

b.) Find a point, (x_0,y_0,z_0), that lies on both planes.

given by the equation $3x + y + 2z = 3$.

a.) Find the angle between the planes. ACUTE

b.) Find a point, (x_0, y_0, z_0), that lies on both planes.

c.) Find ~~a~~ parametric equations for the line where the two planes intersect.

$\vec{n}_1 = \langle 2, -1, 3 \rangle$

$\vec{n}_2 = \langle 3, 1, 2 \rangle$

a) $\cos\theta = \left| \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} \right| = \left| \dfrac{6 - 1 + 6}{\sqrt{4+1+9}\,\sqrt{9+1+4}} \right| = \dfrac{11}{14}$

$\theta = \cos^{-1}\left(\frac{11}{14}\right)$

$4 - (-3) = 7$ ✓

b) $2x - y + 3z = 7$

$3x + y + 2z = 3 \rightarrow 6 + y + 0 = 3$

$y = -3$

$5x + 5z = 10$

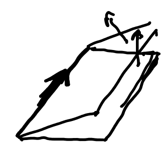
$x + z = 2$ set $z = 0$ then $x = 2$ $P_0(2, -3, 0)$

c) $P_0(2, -3, 0)$

$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 3 & 1 & 2 \end{vmatrix} = \langle -5, +5, 5 \rangle = \vec{v}$

$\approx \langle -1, 1, 1 \rangle$

$\begin{cases} x = 2 - 5t \\ y = -3 + 5t \\ z = 0 + 5t \end{cases}$



Problem 12. Consider the lines $\mathbf{r_1}(t) = \langle 1, 2, 0 \rangle + t\,\langle 2, -2, 2 \rangle$ and $\mathbf{r_2}(v) = \langle 3, 0, 2 \rangle + v\,\langle -2, 2, 0 \rangle$.

a.) Find the point where the two lines intersect.

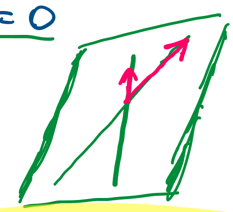
b.) Find an equation of the plane containing both of these lines.

a) $1 + 2t = 3 - 2v \rightarrow 1 + 2 = 3 - 0$

$2 - 2t = 0 + 2v \rightarrow 0 = 2v \rightarrow v = 0$

$0 + 2t = 2 \rightarrow t = 1$

point $P_0(3, 0, 2)$



b) $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 2 \\ -2 & 2 & 0 \end{vmatrix} = \langle -4, -4, 0 \rangle = \vec{n} = \langle -4, -4, 0 \rangle$

$\langle +1\ +1\ 0 \rangle$

$-4(x - 3) - 4(y - 0) + 0(z - 2) = 0$

$4(x - 3) + 4(y - 0) = 0$

$x + y - 3 = 0$

Problem 13. Let $\mathbf{r}(t) = \langle t^2, \frac{t-1}{t^2-1}, \frac{\sin t}{t} \rangle$.

$\frac{\cancel{t-1}}{(\cancel{t-1})(t+1)}$

a.) Find the domain of $\mathbf{r}(t)$.

b.) Find $\lim_{t \to 1} \mathbf{r}(t)$.

$\langle 1, \frac{1}{2}, \frac{\sin 1}{1} \rangle = \langle 1, \frac{1}{2}, \sin 1 \rangle$

a) t^2 : $\mathbb{R}$

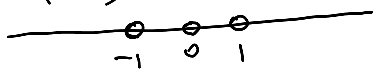
$t - 1$: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

a) $t^2 : \mathbb{R}$

$\frac{t-1}{t^2-1} : (-\infty,-1)\cup(-1,1)\cup(1,\infty)$

$\frac{\sin t}{t} : (-\infty,0)\cup(0,\infty)$

Domain $= (-\infty,-1)\cup(-1,0)\cup(0,1)\cup(1,\infty)$



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Problem 14. Let $\mathbf{r}(t) = \langle \cos(t^2), \sin(t^2), t^2 \rangle$.

a.) Find the velocity and speed of the curve at time $t = \sqrt{\pi}$.

b.) Find $\mathbf{T}(\sqrt{\pi})$, the unit tangent vector, at $t = \sqrt{\pi}$.

c.) Find $\mathbf{a}(t)$, the acceleration vector, at time t.

d.) The length of the curve from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

e.) The curvature of the curve traced out by $\mathbf{r}(t)$ when $t = \sqrt{\pi}$.

a) $\vec{v}(t) = \vec{r}\,'(t) = \langle -2t\sin(t^2), 2t\cos(t^2), 2t \rangle$

$|\vec{v}(t)| = \sqrt{\underbrace{4t^2\sin^2 t^2 + 4t^2\cos t^2}_{4t^2} + 4t^2} = \sqrt{8t^2} = 2t\sqrt{2}$.

$\vec{v}(\sqrt{\pi}) = \langle 0, -2\sqrt{\pi}, 2\sqrt{\pi} \rangle$

$|\vec{v}(\sqrt{\pi})| = 2\sqrt{2\pi}$

b) $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle 0, \frac{-\cancel{2}\sqrt{\pi}}{\cancel{2}\sqrt{2\pi}}, \frac{2\sqrt{\pi}}{2\sqrt{2\pi}} \rangle =$

$= \langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \vec{T}(\sqrt{\pi})$

c)

$\vec{a}(t) = \langle -2\sin t^2 + (-2t)2t\cos t^2, 2\cos t^2 + (2t)2t(-\sin t^2), 2 \rangle$

$= \langle -2\sin t^2 - 4t^2\cos t^2, 2\cos t^2 - 4t^2\sin t^2, 2 \rangle = \langle 4\pi, -2, 2 \rangle$

d)

$\mathcal{L} = \int_0^{\sqrt{2\pi}} |\vec{v}(t)|\,dt = \int_0^{\sqrt{2\pi}} 2t\sqrt{2}\,dt = t^2\sqrt{2}\Big]_0^{\sqrt{2\pi}} =$

$$\mathcal{L} = \int_0 |\vec{v}(t)|\,dt = \int_0 2t\sqrt{2}\,dt = t^2\sqrt{2}\Big|_0 =$$

$$= \sqrt{2}\,(2\pi - 0) = \underline{2\pi\sqrt{2}}$$

e) $$k = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{\sqrt{0 + 64\pi^3 + 64\pi^3}}{\pi\, 16\sqrt{2\pi}} = \frac{\sqrt{2^7\pi^3}}{2^4\sqrt{2\pi}} = \frac{\cancel{2^3}\cancel{\pi}\cancel{\sqrt{2\pi}}}{\cancel{\pi}\,\cancel{2^4}\cancel{\sqrt{2\pi}}} = \frac{1}{2} = \frac{1}{2}$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -2\sqrt{\pi} & 2\sqrt{\pi} \\ 4\pi & -2 & 2 \end{vmatrix} = \langle 0, +8\pi\sqrt{\pi}, 8\pi\sqrt{\pi} \rangle$$

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Problem 15. Find parametric equations for the tangent line to the curve $x = 4\sqrt{t}$, $y = t^2 - 10$, $z = \frac{4}{t}$ at $(8, 6, 1)$. P_0 $t = 4$

$$\vec{v} = \vec{r}\,'(t) = \left\langle 4\frac{1}{2\sqrt{t}}, 2t, -\frac{4}{t^2} \right\rangle$$

$$\vec{r}\,'(4) = \left\langle 1, 8, -\frac{1}{4} \right\rangle$$

$$\begin{cases} x = 8 + t \\ y = 6 + 8t \\ z = 1 - \frac{t}{4} \end{cases}$$

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Problem 16. If $\mathbf{r}'(t) = \langle t, e^t, te^{3t} \rangle$ and $\mathbf{r}(0) = \langle 1, 3, 2 \rangle$, find $\mathbf{r}(t)$.

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$\vec{r}(t) = \int \vec{r}\,'(t)\,dt =$

$= \langle \frac{t^2}{2} + c_1 ,\ e^t + c_2 ,\ \frac{t}{3} e^{3t} - \frac{1}{9} e^{3t} + c_3 \rangle$

t	e^{3t}
1	$\frac{1}{3} e^{3t}$
0	$\frac{1}{9} e^{3t}$

$\vec{r}(0) = \langle c_1, 1 + c_2, -\frac{1}{9} + c_3 \rangle = \langle 1, 3, 2 \rangle$

$c_1 = 1$ $\quad c_2 = 2$ $\quad c_3 = \frac{19}{9}$

$2 + \frac{1}{9}$

$\vec{r}(t) = \langle \frac{t^2}{2} + 1, e^t + 2, \frac{t}{3} e^{3t} - \frac{1}{9} e^{3t} + \frac{19}{9} \rangle$

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Problem 17. Find $\int_0^1 \left(\frac{4t}{t^2+1}\mathbf{j} - \frac{1}{\underbrace{1+t^2}}\mathbf{k} \right) dt$.

$\boxed{u = t^2 + 1}$
$du = 2t\,dt$

$\int_1^2 \frac{2du}{u}\,\hat{j} - \Big[\text{arcant } t\Big]_0^1 \hat{k}$

$2 \ln u \Big]_1^2 \hat{j} - \left[\frac{\pi}{4} - 0 \right] \hat{k}$

$2(\ln 2 - \ln 1)\hat{j} - \frac{\pi}{4}\hat{k} = 2\ln 2\,\hat{j} - \frac{\pi}{4}\hat{k}$

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Problem 18. Given the curves $\mathbf{r}_1(t) = \langle 3t, t^2, t^3\rangle$ and $\mathbf{r}_2(v) = \langle \sin v, \sin(2v), 6v\rangle$ intersect at the origin, find the angle of intersection. $t=0$ $v=0$ $(0,0,0)$

$$\vec{r}_1{}'(t) = \langle 3, 2t, 3t^2 \rangle \quad \langle 3, 0, 0 \rangle$$

$$\vec{r}_2{}'(v) = \langle \cos v, 2\cos(2v), 6 \rangle \quad \langle 1, 2, 6 \rangle$$

$$\theta = \cos^{-1} \frac{\vec{r}_1{}'(0) \cdot \vec{r}_2{}'(0)}{|\vec{r}_1{}'(0)|\,|\vec{r}_2{}'(0)|} = \cos^{-1} \frac{\cancel{3}}{\cancel{3}\sqrt{1+4+36}} =$$

$$= \cos^{-1} \frac{1}{\sqrt{41}} .$$

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