Wir 11: Chapter 16: 16.1-16.9

Problem 1. Evaluate $\int_C y\,ds$, where C is parameterized by $\mathbf{r}(t) = \langle\, t,\ t^3 \rangle$, $0 \le t \le 1$.

$$ds = \sqrt{x'^2 + y'^2 + z'^2}\ dt$$

$$\int_0^1 t^3 \sqrt{1^2 + (3t^2)^2}\ dt = \int_0^1 t^3 \sqrt{1 + 9t^4}\ dt =$$

$u = 1 + 9t^4$

$du = 36 t^3 dt$

$$= \int_1^{10} \frac{1}{36} u^{1/2}\ du = \frac{1}{\cancel{36}_{18}} \frac{\cancel{2}}{3} u^{3/2} \Big|_1^{10} = \frac{1}{54}\left(10^{3/2} - 1\right)$$

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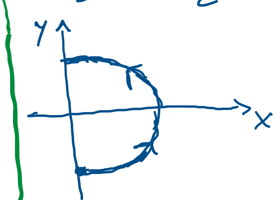
Problem 2. Find $\int_C x\,ds$, where C is the right half of the circle $x^2 + y^2 = 4$, oriented counter-clockwise.

$x = 2\cos\theta$, $y = 2\sin\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



$$\int_{-\pi/2}^{+\pi/2} 2\cos\theta \underbrace{\overbrace{\sqrt{(-2\sin\theta)^2 + (2\cos\theta)^2}}^{=2}}_{\sqrt{4}}\ d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} 4\cos\theta\ d\theta = \Big[4\sin\theta\Big]_{-\pi/2}^{\pi/2} = 4(1 - -1) = 8.$$

Problem 3. Evaluate $\int_C z\,dx + (xy)dy$, where C is the line segment from $(-1, 1, 0)$ to $(1, 2, 0)$.

$$\int_C \vec{F} \cdot d\vec{r} \qquad \vec{F} = \langle z, xy, 0 \rangle$$

$\vec{r}(t) = \langle -1, 1, 0 \rangle + t\langle 2, 1, 0 \rangle =$

$= \langle -1 + 2t, 1 + t, 0 \rangle$

$0 \le t \le 1$

$dy = y'(t)dt =$

$= 1\,dt$

$$\int_0^1 0\,dx + (-1 + 2t)(1 + t)\,dt =$$

$$= \int -1 - t + 2t + 2t^2\ dt = \int_0^1 (-1 + t + 2t^2)\,dt = \left[-t + \frac{t^2}{2} + 2\frac{t^3}{3}\right]_0^1 =$$

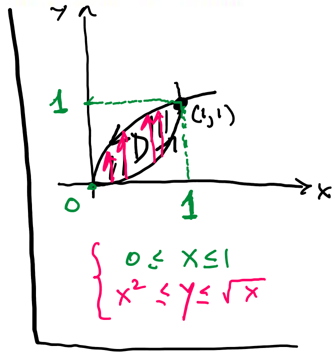
$= \frac{-6+3+4}{} = \frac{1}{}.$

$$= \int_0 -1 - t + 2t + 2t^2 \, dt = \int_0 (-1 + t + 2t^2)\, dt = \Big[\quad \Big]_0$$

$$= -1 + \frac{1}{2} + \frac{2}{3} = \frac{-6+3+4}{6} = \frac{1}{6}.$$

Problem 4. Find $\displaystyle\int_C \underbrace{(3y + 7e^{\sqrt{x}})}_{P}dx + \underbrace{(8x + 9\cos(y^2))}_{Q}dy$, where C is the boundary of the region enclosed by $y = x^2$ and $x = y^2$.

Green's Theorem $\displaystyle = \iint_D (Q_x - P_y)\, dA$



$Q_x = 8 \qquad P_y = 3$

$$\iint_D 5 \, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} 5 \, dy\, dx = \int_0^1 5(\sqrt{x} - x^2)\, dx =$$

$$= 5\left[\frac{2}{3} x^{3/2} - \frac{x^3}{3} \right]_0^1 = 5\left(\frac{2}{3} - \frac{1}{3} \right) = \frac{5}{3}.$$

$dy = y'(t)\, dt$

Problem 5. Evaluate $\displaystyle\int_C (xy)\, dx + (x - y)dy$, where C is the line segment from $(1, 1)$ to $(2, 0)$ and then from $(2, 0)$ to $(3, 5)$.

$\vec{r}_1(t) = \langle 1+t, 1-t \rangle \qquad 0 \le t \le 1$

$\vec{r}_2(t) = \langle 2+t, 5t \rangle \qquad 0 \le t \le 1$

$$\int_C = \int_{C_1} + \int_{C_2} = \int_0^1 (1-t^2)\, dt - (2t)\, dt + \int_0^1 (10t + 5t^2)\, dt + (2-4t)5\, dt =$$

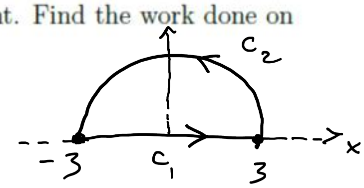
$10t + 5t^2 + 10 - 20t$

$\boxed{5t^2 - 10t + 10}$

$$= \left[t - \frac{t^3}{3} - t^2 \right]_0^1 + \left[5\frac{t^3}{3} - 5t^2 + 10t \right]_0^1 =$$

$$\cancel{1} - \frac{1}{3} \cancel{-1} + \frac{5}{3} - 5 + 10 = \frac{4}{3} + 5 = \frac{4+15}{3} = \frac{19}{3}.$$

Problem 6. A particle starts at the point $(-3, 0)$, moves along the x-axis to the point $(3, 0)$, then along the semicircle $y = \sqrt{9 - x^2}$, then back to the starting point. Find the work done on this particle by the force field $\mathbf{F} = \langle 3x, x^3 + 3xy^2 \rangle$.



$$\text{Work} = \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} + \int_{C_2} =$$

$[0,1]$ $\vec{r}_1(t) = \langle -3 + 6t, 0 \rangle$

$\vec{r}_2(t) = \langle 3\cos t, 3\sin t \rangle$, $0 \le t \le \pi$

$$= \int_0^1 \langle -9 + 18t, (-3+6t)^3 + 0 \rangle \cdot \langle 6, 0 \rangle \, dt + $$

$$+ \int_0^\pi \langle 9\cos t, 27\cos^3 t + 81 \cos t \sin^2 t \rangle \cdot \langle -3\sin t, 3\cos t \rangle dt$$

$$= 6\int_0^1 (-9 + 18t)\, dt + \int_0^\pi -27\cos t \sin t + 81\cos^4 t + 243\cos^2 t \sin^2 t \; dt =$$

$$= 6\left[-9t + 9t^2\right]_0^1 - 27 \tfrac{1}{2}\sin^2 t\Big]_0^\pi + 81 \int \tfrac{1}{2}(1 + \cos 2t)\, \tfrac{1}{2}(1 + \cos 2t)\, dt +$$

$$\underbrace{+ 243 \int \tfrac{1}{4}\left(1 - \cos^2 2t\right) dt} \qquad \underbrace{\tfrac{81}{4} \int 1 + 2\cos 2t + \underbrace{\cos^2 2t}_{\tfrac{1}{2}(1 + \cos 4t)}}$$

$$= 6\cancel{\left[-9 + 9 - 0\right]} - \cancel{\tfrac{27}{2} \cdot 0} + \tfrac{81}{4}\left[t + \cancel{\sin 2t} + \underbrace{\tfrac{1}{2}\left(t + \cancel{\tfrac{1}{4}\sin 4t}\right)}\right]_0^\pi +$$

$$+ \frac{243}{4}\left[t - \frac{1}{2}\left(t + \cancel{\tfrac{1}{4}\sin 4t}\right)\right]_0^\pi = \frac{81}{4}\left(\pi + \frac{\pi}{2}\right) + \frac{243}{4}\left[\pi - \frac{\pi}{2}\right] =$$

$$= \frac{81}{4}\,\frac{3}{2}\pi + \frac{243}{4}\,\frac{\pi}{2} = \left(\frac{243 + 243}{8}\right)\pi =$$

$$= \frac{486}{8}\pi = \frac{243}{4}\pi$$

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→ conservative b/c $Q_x = P_y = 0$

Problem 7. Find the work done by the force field $\mathbf{F} = \langle x^2, y^2 \rangle$ in moving a particle along the arc of the parabola $y = 2x^2$ from the point $(-2, 8)$ to $(1, 2)$.

$$\int_C \vec{F} \cdot d\vec{r} = f(1,2) - f(-2,8) =$$

$$\boxed{\langle x^2, y^2 \rangle = \vec{\nabla} f}$$

$$\boxed{f(x,y) = \frac{x^3}{3} + \frac{y^3}{3}}$$

64
8
512

$$= \frac{1^3 + 2^3}{} - \frac{(-2)^3 + 8^3}{} = \frac{9}{3} - \left\{\frac{-8 + 512}{3}\right\}$$

$$= \frac{1^3+2^3}{3} - \frac{(-2)^3+8^3}{3} = \frac{9}{3} - \left\{\frac{-8+512}{3}\right\}$$

$$= -\frac{\cancel{495}}{\cancel{3}} = -165$$

$512 - 17 = 495$

$\operatorname{curl}(\vec{F}) = (\underbrace{Q_x - P_y}_{0})\hat{k}$

→ conservative b/c $\operatorname{curl}\vec{F} = \vec{0}$

Problem 8. Given $\mathbf{F} = \langle 4xe^z, \cos(y), 2x^2e^z\rangle$ and $\mathbf{r}(t) = \langle \sin(t), t, \cos(t)\rangle$, compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $0 \le t \le \frac{\pi}{2}$.

$\vec{F} = \vec{\nabla} f$ $\quad \boxed{f = 2x^2e^z + \sin y}$

$\vec{r}\left(\frac{\pi}{2}\right) = \langle 1, \frac{\pi}{2}, 0\rangle$

$\vec{r}(0) = \langle 0, 0, 1\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = f\left(\vec{r}\left(\tfrac{\pi}{2}\right)\right) - f\left(\vec{r}(0)\right) =$$

$$= f\left(1, \tfrac{\pi}{2}, 0\right) - f(0,0,1) = [2(1)(1)+1] - 0 = 3$$

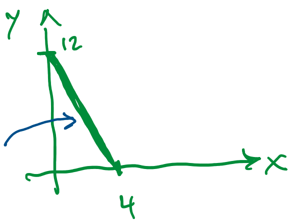
same → $\text{Area} = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$

Problem 9. Find the surface area of the part of the plane $6x + 2y + 8z = 24$ in the first octant.

$\text{Area} = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$

$$z = \frac{24}{8} - \frac{6x}{8} - \frac{2y}{8} = 3 - \frac{3}{4}x - \frac{1}{4}y$$

$$y = \frac{24 - 6x}{2} = 12 - 3x$$



$$\int_0^4 \int_0^{12-3x} \sqrt{1 + \left(-\tfrac{3}{4}\right)^2 + \left(-\tfrac{1}{4}\right)^2}\, dy\, dx =$$

$$= \sqrt{1 + \frac{9}{16} + \frac{1}{16}} \int_0^4 [12 - 3x - 0]\, dx = \sqrt{\frac{\cancel{26}\,13}{\cancel{16}\,8}} \left[12x - 3\frac{x^2}{2}\right]_0^4 =$$

$$= \sqrt{\frac{13}{8}}\left(48 - \frac{3}{\cancel{2}} \cdot \cancel{16}\,8 - 0\right) = 24\sqrt{\frac{13}{8}} = \frac{\cancel{24}\,12\sqrt{13}}{\cancel{8}\sqrt{2}} =$$

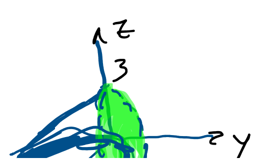
$$= 12\sqrt{\frac{13}{2}}.$$

Problem 10. Find the surface area of the part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

$\text{Area} = \iint_D \sqrt{1 + f_y^2 + f_z^2}\, dA$

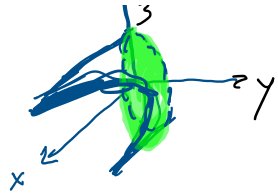
$\sqrt{1 + (2y)^2 + (2z)^2} =$

$= \sqrt{1 + 4(y^2 + z^2)}$



Area $= \iint_D \sqrt{1+f_y^2+f_z^2}\, dA$

$= \sqrt{1+4(y^2+z^2)}$



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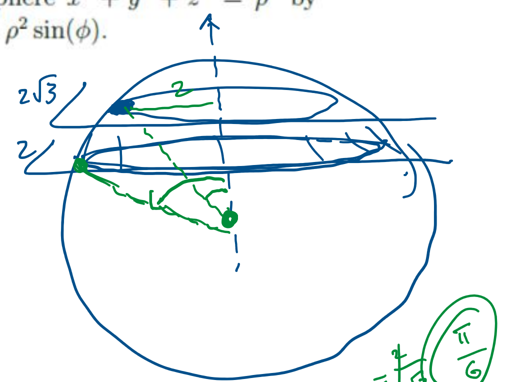
$$\int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\; r\,dr\,d\theta = 2\pi \cdot \frac{1}{8}\cdot\frac{2}{3}\left[(1+4r^2)^{3/2}\right]_0^3 =$$

$$= \frac{\pi}{6}\left[37^{3/2} - 1\right]$$

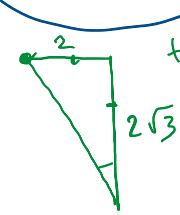
Problem 11. Set up but do not evaluate an integral which gives is the correct set up in order to evaluate $\iint_S yz\,dS$ where S is the part of the sphere $x^2+y^2+z^2 = 16$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$. Note: If we parameterize the sphere $x^2+y^2+z^2 = \rho^2$ by $\mathbf{r}(\theta,\phi) = \langle \rho\sin(\phi)\cos(\theta),\ \rho\sin(\phi)\sin(\theta),\ \rho\cos(\phi)\rangle$, then $|\mathbf{r}_\theta \times \mathbf{r}_\phi| = \rho^2\sin(\phi)$.

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/3} 16\sin\varphi\cos\varphi\sin\theta\; 16\sin\varphi\; d\varphi\, d\theta$$

$0 \to \theta$, $\frac{\pi}{6} \to \varphi$



$\tan\varphi = \frac{2}{2\sqrt{3}}$ $\frac{\pi}{6}$



$z = 4\cos\varphi$

Problem 12. Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = \langle\, y,\ x,\ z\rangle$ and S is the part of the paraboloid $z = x^2+y^2$ between the planes $z = 1$ and $z = 4$.

$$\iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\, dA$$

$\vec{r}(u,v) = \langle u, v, u^2+v^2\rangle$

$\vec{F}(\vec{r}(u,v)) = \langle v, u, u^2+v^2\rangle$

$r\cos\theta = u$

$r\sin\theta = v$

$$\vec{r}_u\times\vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = \langle +2u, +2v, -1\rangle$$

$$\iint \big(\underbrace{2uv+2uv}_{4uv} - u^2 - v^2\big)\, dA$$

Ring

$6\sin\theta\cos\theta$

$$\int_0^{2\pi}\int_1^2 (4\sin\theta\cos\theta - r^2)\, r\, dr\, d\theta$$

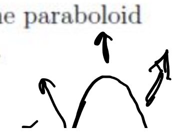
$4r\sin\theta\cos\theta - r^3$

$$2r^2\sin\theta\cos\theta - \frac{r^4}{4}\Big|_1^2 = 8\sin\theta\cos\theta - \frac{16}{4} - 2\sin\theta\cos\theta + \frac{1}{4} =$$

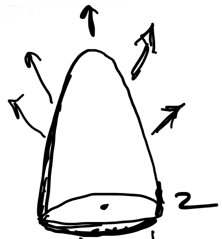
$$\left[3\sin^2\theta - \frac{15}{4}\theta\right]_0^{2\pi} = -\frac{15}{4}\cdot 2\pi = -\frac{15}{2}\pi$$

Problem 13. Find the flux of $\mathbf{F} = \langle\, x,\ y,\ -z\rangle$ across S, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that is above the xy-plane. Use the positive (outward) orientation.

$\vec{r}(u,v) = \langle u, v, 4-u^2-v^2\rangle$



$z = \underbrace{4 - x^2 - y^2}$ that is above the xy-plane. Use the positive (outward) orientation.



$\vec{r}(u,v) = \langle u, v, 4 - u^2 - v^2 \rangle$

$\vec{F}(\vec{r}(u,v)) = \langle u, v, -4 + u^2 + v^2 \rangle$ dot product $\langle 2u, 2v, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = \langle 2u, 2v, 1 \rangle$

$2u^2 + 2v^2 - 4 + u^2 + v^2 = -4 + 3(u^2 + v^2)$

$$\int_0^{2\pi}\int_0^2 (-4 + 3r^2)\, r \, dr\, d\theta = 2\pi\left[-2r^2 + \frac{3}{4}r^4\right]_0^2 =$$

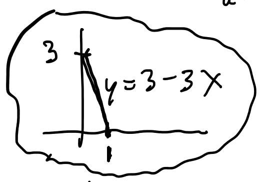
$-4r + 3r^3$

$$= 2\pi\left[-8 + \frac{3}{\cancel{4}_1}\cancel{16}^{4} - 0\right] = 8\pi$$

Problem 14. Use Stokes' Theorem to set up but **not evaluate** $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle xz, 2xy, 3xy \rangle$ C is the boundary curve of part of plane $\underbrace{3x + y + z = 3}$ in first octant.

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{vmatrix} = \langle 3, +1, 1 \rangle$



$\underbrace{\vec{r}(u,v) = \langle u, v, 3 - 3u - v \rangle}$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xz & 2xy & 3xy \end{vmatrix} = \langle 3x - 0, -(3y - x), 2y - 0 \rangle$$

$\underbrace{\langle 3u, -3v + u, 2v \rangle}$ $\underbrace{\langle 3, 1, 1 \rangle}$

$$\int_0^1 \int_0^{3-3u} (9u - 3v + u + 2v)\, dv\, du \quad \ldots$$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} =$ $\langle -1, 0, 1 \rangle$

Problem 15. Set up but do not evaluate the integral which is the result of using Stokes' Theorem to find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle 2xz, 4x^2, 5y^2 \rangle$ and C is curve of intersection of the plane $\underline{z = x + 4}$ and the cylinder $x^2 + y^2 = \underline{4}$, oriented counterclockwise when viewed from above.

$\langle u, v, u + 4 \rangle$

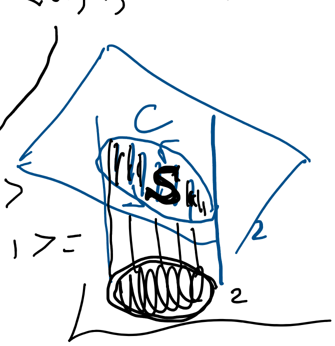
$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xz & 4x^2 & 5y^2 \end{vmatrix} =$$

$= \langle 10y - 0, -(0 - 2x), 8x - 0 \rangle$

$\langle 10v, 2u, 8u \rangle \langle -1, 0, 1 \rangle =$

$= -10v + 8u$



$$\iint_{D_2} (-10v + 8u)\, dA =$$

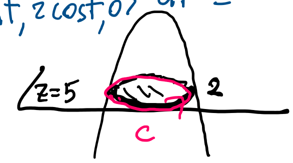
$\int^{2\pi} \int^{2} (\ldots) r\, dr\, d\theta$

D_2

$$= \int_0^{2\pi}\int_0^{2} (8r\cos\theta - 10r\sin\theta)\, r\,dr\,d\theta$$

Problem 16. Use Stokes' Theorem evaluate $\iint_S \operatorname{curl} \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = \langle x^2\sin(z-5), y^2, xy\rangle$ and S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle \underbrace{4\cos^2 t\cdot 0}_{0}, 4\sin^2 t, 4\cos t\sin t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\, dt =$$



$$= \int_0^{2\pi} 8\sin^2 t\cos t\, dt = \frac{8}{3}\sin^3 t \Big|_0^{2\pi} = 0$$

$\vec{r}(t) = \langle 2\cos t, 2\sin t, 5\rangle$

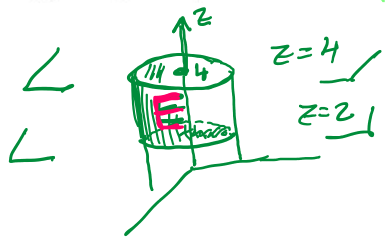
$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (\operatorname{div}\vec{F})\, dV$$

Problem 17. Using the The Divergence Theorem to evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$\mathbf{F} = \left\langle 4x,\ \sin(e^z),\ \sqrt{x^3+y^2}\right\rangle$ and S is the surface bounded by $x^2+y^2=4$, $z=2$, $z=4$.

of the solid E

$$\iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E 4\, dV = 4\boxed{\iiint_E dV} =$$



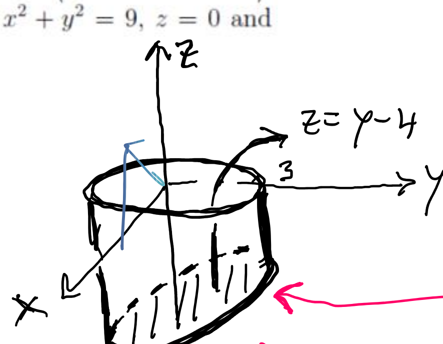
$\operatorname{div}\vec{F} = 4 + 0 + 0$

$= 4\cdot \text{Volume of } E = 4\pi(2^2)2 = \underline{32\pi}$

$\operatorname{div}\vec{F} = 0 + 0 + 2 = 2$

Problem 18. Using the The Divergence Theorem, find the flux of $\mathbf{F} = \left\langle ye^{z^2},\ ze^x,\ 2z+8\right\rangle$ across S, where S is the surface of the solid bounded by the cylinder $x^2+y^2=9$, $z=0$ and $z = y-4$. NOTE THAT $z \le 0$ because $y \le 1$.

$$\iiint_E 2\, dV = (-2)\int_0^{2\pi}\int_0^{3}\int_0^{r\sin\theta - 4} dz\, r\, dr\, d\theta =$$



The negative sign because the inner integral should go from $r\sin\theta - 4$ to zero.

$(r\sin\theta - 4 - 0)r$

$r^3 \qquad r^3\sin\theta - 2r^2 \Big|^3 =$

$$\int_0^3 \left[r^2 \sin\theta - 4r\right] dr = \frac{r^3}{3}\sin\theta - 2r^2 \Big|_0^3 =$$

$$= \int_0^{2\pi} 9\sin\theta - 18 \; d\theta$$

$$\left[-9\cancel{\cos\theta} - 18\theta\right]_0^{2\pi} = -18\cdot 2\pi = -36\pi.$$

Answer: $(-2)(-36\pi) = 72\pi$