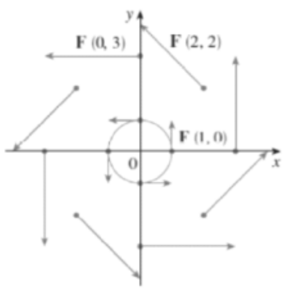
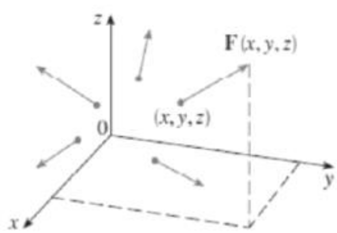
Wir 10: Sections 16.1, 16.2, 16.3, 16.4, 16.5

Section 16.1

Definition: A **vector field** in two dimension is a function **F** that assigns to each point (x, y) in $D \subset \mathbb{R}^2$ a two dimensional vector, $\mathbf{F}(x, y)$.
In two dimension, the vector field lies entirely in the xy plane.
Here is a vector field in $\mathbb{R}^2$:



Definition: A **vector field** in three dimension is a function **F** that assigns to each point (x, y, z) in $D \subset \mathbb{R}^3$ a three dimensional vector, $\mathbf{F}(x, y, z)$.
In three dimension, the vector field is in space.
Here is a vector field in $\mathbb{R}^3$:

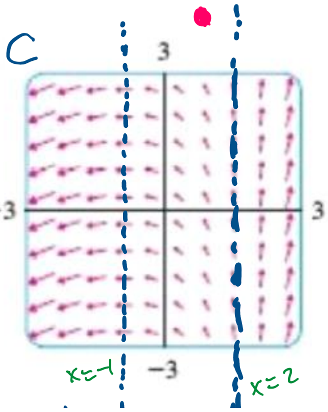


In order to match **F** with its vector field, choose a several points, (x, y), in each quadrant, and look at the *direction* of $\mathbf{F}(x, y)$. To narrow down further, look at the behavior of the components. Often times, it is a process of elimination.

Problem 1. Match each vector field equation with its graph:

a) $\boldsymbol{F}(x, y) = < y, x >$
b) $\boldsymbol{F}(x, y) = < 1, \sin y >$
c) $\boldsymbol{F}(x, y) = < x - 2, x + 1 >$
d) $\boldsymbol{F}(x, y) = < y, \frac{1}{x} >$

I C



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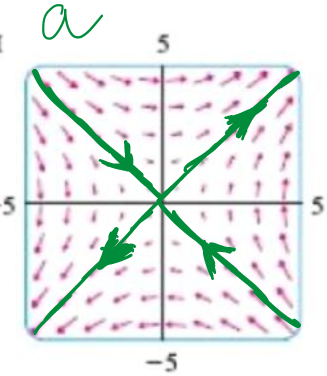
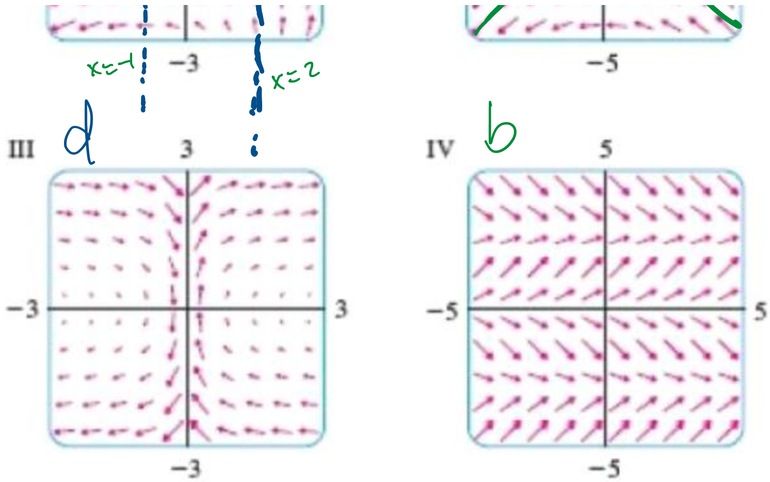
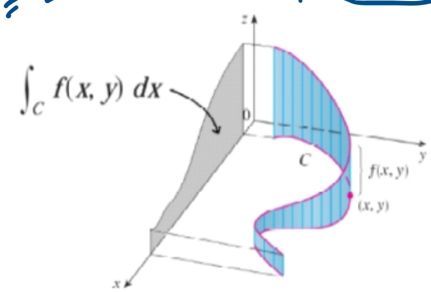




Section 16.2

Definition: If f is defined on a smooth curve C defined as $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $a \leq t \leq b$, then the **line integral of f along C** is

$$\int_C f(x,y)ds = \int_a^b (f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b (f(x(t),y(t))|\mathbf{r}'(t)|\, dt$$



In order to find a line integral along a curce C, we must first parameteterize the curve. Sometimes, the parameterization will be given explicitly, other times you must parameterize the curve.

Problem 2. Evaluate $\int_C (2x+y)ds$, where C is defined as $\mathbf{r}(t) = \langle 2+t, 3-t\rangle$, $0 \leq t \leq 1$.

(2,3) to (3,2)

$$\int_0^1 [2(2+t)+3-t]\sqrt{1^2+(-1)^2}\, dt = \sqrt{2}\int_0^1 \underbrace{4+2t+3-t}_{(7+t)}\, dt$$

$$7t + \frac{t^2}{2}\Big|_0^1 = 7 + \frac{1}{2} = 7.5 = \frac{15}{2}$$

$$\text{Answer} = \frac{15\sqrt{2}}{2}$$

Problem 3. Set up but do not evaluate $\int_C (2x + x^2 y)ds$, where C is the arc of the curve $y = x^2$ from $(1, 1)$ to $(2, 4)$ **using two different parameterizations.**

$\vec{r}_1(t) = \langle t, t^2 \rangle$

$1 \le t \le 2$

$$\int_1^2 \left[2t + t^2(t^2)\right]\sqrt{1^2 + (2t)^2}\, dt$$

$\vec{r}_2(t) = \langle 1+t, (1+t)^2 \rangle$

$0 \le t \le 1$

$$\int_0^1 \left[2(1+t) + (1+t)^2(1+t)^2\right]\sqrt{1 + [2(1+t)]^2}\, dt$$

Problem 4. Evaluate $\int_C (x^2 + y)\, ds$ where C consists of the line segment from the point $\boxed{(1, 4)}$ to $(3, -1)$.

$\vec{r}(t) = \langle 1, 4 \rangle + t\langle 2, -5 \rangle =$

$= \langle 1 + 2t, 4 - 5t \rangle$

$0 \le t \le 1$

$$\int_0^1 \left[(1+2t)^2 + (4-5t)\right]\sqrt{2^2 + (-5)^2}\, dt =$$

$[1 + 4t + 4t^2 + 4 - 5t]\sqrt{4+25}$

$(4t^2 - t + 5)\sqrt{29}$

$$= \sqrt{29}\int_0^1 (4t^2 - t + 5)\, dt =$$

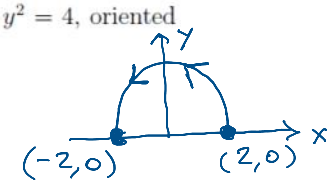
$$= \sqrt{29}\left[4\frac{t^3}{3} - \frac{t^2}{2} + 5t\right]_0^1 = \sqrt{29}\left(\frac{4}{3} - \frac{1}{2} + 5\right) = \frac{35}{6}\sqrt{29}$$

$\frac{8 - 3 + 30}{6}$

Problem 5. Evaluate $\int_C (x + y)ds$, where C is the top half of the circle $x^2 + y^2 = 4$, oriented counterclockwise.

y

Problem 5. Evaluate $\int_C (x+y)ds$, where C is the top half of the circle $x^2+y^2=4$, oriented counterclockwise.



$$\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$$

$$\int_0^{\pi} (2\cos t + 2\sin t)\underbrace{\sqrt{(-2\sin t)^2+(2\cos t)^2}}\,dt =$$

$$\sqrt{4\sin^2+4\cos^2}$$
$$\sqrt{4(\underbrace{\sin^2+\cos^2}_{1})} = 2$$

$$= 2\left[2\sin t - 2\cos t\right]_0^{\pi} =$$

$$= 2\cdot 2\left[0 - \cos\pi - -\cos 0\right] =$$

$$= 2\cdot 2\left[1+1\right] = 4\cdot 2 = 8$$

Problem 6. Set up but do not evaluate $\int_C (2+x^2y)ds$, where C is the arc of the curve $x=y^2$ from $(1,-1)$ to $(4,2)$ and then along the line segment from the point $(4,2)$ to the point $(3,7)$.

$C = C_1 \cup C_2$

$C_1 : \langle t^2, t\rangle \quad -1 \le t \le 2$

$C_2 : \langle 4,2\rangle + t\langle -1,5\rangle \quad 0 \le t \le 1$

$$\int_C = \int_{C_1} + \int_{C_2} = \int_{-1}^{2} (2+t^4\cdot t)\sqrt{(2t)^2+1}\,dt + \int_0^1 \left(2+(4-t)^2(2+5t)\right)\sqrt{(-1)^2+5^2}\,dt =$$

$$= \int_{-1}^{2} (2+t^5)\sqrt{4t^2+1}\,dt + \int_0^1 \left[2+(4-t)^2(2+5t)\right]\sqrt{26}\,dt$$

#

Line Integrals over vector fields: Suppose now are moving a particle along a curve C through a vector (force) field, $\mathbf{F}$. We define the **line integral of F along C** to be a vector (force) field, **F**. We define the **line integral of F along C** to be

$$\text{Work} = \int_c \mathbf{F}\cdot d\mathbf{r} = \int_a^b (\mathbf{F}(\mathbf{r}(t))\bullet\mathbf{r}'(t)dt$$

Problem 7. Find $\int_c \mathbf{F}\cdot\mathbf{r}$, where C is defined by $\mathbf{r}(\mathbf{t}) = \langle t, t^2, t^4\rangle$, $0 \le t \le 1$, and $\mathbf{F}(x,y,z) = \langle x, z^2, -4y\rangle$.

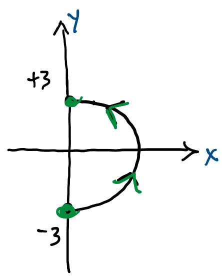
$$\left.\begin{array}{l}\to \vec{F}(\vec{r}(t)) = < t,\ t^8,\ -4t^2 > \\ \to \vec{r}'(t) = < 1,\ 2t,\ 4t^3 >\end{array}\right\}\ \underline{\text{dot}} = t + 2t^9 - 16t^5$$

$$\int_0^1 (t + 2t^9 - 16t^5)\, dt = \frac{1}{2}t^2 + \frac{2}{10}t^{10} - \frac{16}{6}t^6\Big|_0^1 =$$

$$= \frac{1}{2} + \frac{1}{5} - \frac{8}{3} = \frac{15+6-80}{30} = \frac{-59}{30} \ \#$$

Problem 8. Find the work done by the force field $\mathbf{F}(x,y) = \langle x^2, xy\rangle$ in moving an object counterclockwise around the right half of the circle $x^2 + y^2 = 9$.

$\vec{r}(t) = < 3\cos t,\ 3\sin t >$ $\quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$



$\vec{F} = < 9\cos^2 t,\ 9\cos t\sin t > = \vec{F}(\vec{r}(t))$

$\vec{r}' = < -3\sin t,\ 3\cos t > = \vec{r}'(t)$

$\vec{F}\cdot\vec{r}' = -27\cos^2 t\sin t + 27\cos^2 t\sin t = 0$

Work = 0

Definition: Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ for $a \le t \le b$. The **line integral of f along** C **with respect to** x is $\int_C f(x,y)dx = \int_a^b (f(x(t), y(t))\ x'(t)\ dt.$

$dx = x'(t)\,dt$

The **line integral of f along** C **with respect to** y is $\int_C f(x,y)dy = \int_a^b (f(x(t), y(t))\ y'(t)\ dt$

$dy = y'(t)\,dt$

Problem 9. $\int_C y dx + x^2 dy$, where C is decribed by $\mathbf{r}(t) = \langle 3e^t, e^{2t}\rangle$, $0 \le t \le 1$.

$$\int_0^1 e^{2t}\, 3e^t\, dt + 9e^{2t}\, 2e^{2t}\, dt = \int_0^1 (3e^{3t} + 18e^{4t})\, dt =$$

$1 + 9/2 = \frac{2+9}{2} = \frac{11}{2}$

$$e^{3t} + \cancel{18}^{9} \frac{1}{\cancel{4}_2} e^{4t} \Big|_0^1 = e^3 + \frac{9}{2}e^4 - \left(e^0 + \frac{9}{2}e^0\right) =$$

$$= e^3 + \frac{9}{2}e^4 - \frac{11}{2}.$$

$\vec{F} = \langle y, x^2 \rangle$

$\vec{F} = \langle x, y \rangle$

Problem 10. Evaluate $\int_C x dx + y dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.

$\vec{r}(t) = \langle 4 - t^2, t \rangle$

$-3 \le t \le 1$

$$\int_{-3}^1 (4 - t^2)(-2t)\,dt + t\,(1)\,dt = \int_{-3}^1 (-8t + 2t^3 + t)\,dt =$$

$$= \int_{-3}^1 (2t^3 - 7t)\,dt = \cancel{2}\frac{t^4}{\cancel{4}_2} - 7\frac{t^2}{2}\Big|_{-3}^1 =$$

$$= \left(\frac{1}{2} - \frac{7}{2}\right) - \left(\frac{(-3)^4}{2} - \frac{7}{2}(-3)^2\right) = \left(-\frac{6}{2}\right) - \left(\frac{81}{2} - \frac{63}{2}\right) =$$

$$= -3 - \left(\frac{18}{2}\right) = -3 - 9 = -12$$

#

81
63
18

$\vec{F} = \langle x+y, y-x, z \rangle$

18

$\vec{F} = \langle x+y, y-x, z\rangle$

Problem 11. Evaluate $\int_C (x+y)dz + (y-x)dy + zdx$ where C is decribed by $x = t^4$, $y = t^3$, $z = t^2$, $0 \le t \le 1$.

$$\int_0^1 (t^4 + t^3)\, 2t\, dt + (t^3 - t^4)\, 3t^2 dt + t^2\, 4t^3 dt =$$

$$2t^5 + 2t^4 + 3t^5 - 3t^6 + 4t^5$$

$$\int_0^1 (-3t^6 + 9t^5 + 2t^4)\, dt = -\frac{3}{7} + \frac{9}{6} + \frac{2}{5} = \frac{-90 + 315 + 84}{210} = \frac{\cancel{309}\ 103}{\cancel{210}\ 70} =$$

399
90

35
9
315

$$= \frac{103}{70}$$

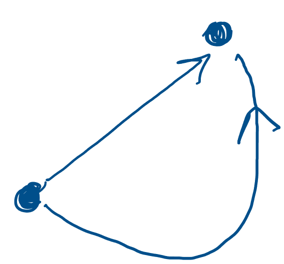
Section 16.3

In section 16.2, we learned how to find a line integral over a vector field **F** along a curve C that is parametrized by $\mathbf{r}(t)$, $a \le t \le b$.

Problem 1. Suppose we are moving a particle from the point $(0,0)$ to the point $(2,4)$ in a force field $\mathbf{F}(x,y) = \langle y^2, x\rangle$. Find $\int_c \mathbf{F} \cdot d\mathbf{r}$,where

a.) The particle travels along the line segment from $(0,0)$ to $(2,4)$. $\vec{r}(t) = \langle 2t, 4t\rangle \quad 0 \le t \le 1$

b.) The particle travels along the curve $y = x^2$ from $(0,0)$ to $(2,4)$. $\vec{r}(t) = \langle t, t^2\rangle \quad 0 \le t \le 2$



a) $\int_0^1 \langle 16t^2, 2t\rangle \cdot \langle 2, 4\rangle\, dt = \int_0^1 (32t^2 + 8t)\, dt = \frac{32}{3} + \frac{\cancel{8}\,4}{\cancel{2}\,1} = \frac{44}{3}$

b) $\int_0^2 \langle t^4, t\rangle \cdot \langle 1, 2t\rangle\, dt = \int_0^2 (t^4 + 2t^2)\, dt = \left.\frac{t^5}{5} + \frac{2}{3}t^3\right]_0^2 =$

$$= \frac{32}{5} + \frac{16}{3} = \frac{96 + 80}{15} = \frac{176}{15}$$

Note: Although the end points are the same, the value of the line integral is **different** because the **paths** are different. In this section, we will learn under what conditions the line integral is independent of the path taken.

Definition: If $\mathbf{F}$ is a continuous vector field, we say that $\int_c \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if and only if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 with the same starting and ending points. In other words, the line integral is the same **no matter what path** you travel on as long as the endpoints are the same.

Definition: A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar funtion f, that is there exists a function f so that $\underline{\underline{\mathbf{F} = \nabla f}}$. We call f the **potential function**.

Problem 2. Consider $f(x, y) = x^2y - y^3$. Find the gradient and explain why it is conservative. What is the potential function?

$\vec{\nabla} f = \langle 2xy, x^2 - 3y^2 \rangle$ is conservative because a gradient

potential function $f = x^2y - y^3$ –

Recall the Fundamental Theorem of Calculus tells us that $\int_a^b f'(x)dx = f(b) - f(a)$.

Since $\nabla f = \langle f_x, f_y \rangle$, we can think of the potential function, f, as some sort of antiderivative of ∇f. Hence $\int \mathbf{F} \cdot d\mathbf{r} = \int \nabla f \cdot d\mathbf{r}$.

Fundamental Theorem for Line Integrals: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let $\mathbf{F}$ be a conservative vector field. Let f be a differentiable function of two or three variables whose gradient vector, ∇f, is continuous on C. Then

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Note: **Line integrals of conservative vectors fields are independent of path** because in a conservative vector field, the line integral is computed by only using the **endpoints** of the domain! Therefore, if we are in a conservative vector field, the line integral along a curve C in that vector field will be the same **no matter what curve we travel across** that connects the endpoints together. **WHICH MEANS WE DON'T EVEN NEED TO PARAMETERIZE THE CURVE!**

Question: How do we determine if a vector field is conservative, and if so, how do we find the potential function? The 'test for conservative' we use depends on whether $\mathbf{F}$ is in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Theorem: $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = P\mathbf{i} + Q\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.
Note: This above criteria to determine if a vector field is conservative works only for $\mathbb{R}^2$.

Problem 3. Is $\mathbf{F}(x, y) = \langle \overset{P}{3x^2 - 4y}, \overset{Q}{4y^2 - 2x} \rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

$\vec{F}$ conservative (in 2 dim) $\Leftrightarrow Q_x = P_y$.

$Q_x = -2$ $\qquad P_y = -4$

$\vec{F}$ not conservative, no potential fnctn.

Problem 4. Is $\mathbf{F}(x, y) = \langle \overset{P}{x + y}, \overset{Q}{x - 2} \rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

$Q_x - P_y = 1 - 1 = 0$ $\qquad$ $\vec{F}$ is conservative

$= \langle x+y, x-2 \rangle$

$Q_x - P_y = 1 \quad 1 - 0$

$\langle \underbrace{f_x}, \underline{f_y} \rangle = \langle \underbrace{x+y}, \underline{x-2} \rangle$

$\frac{\partial f}{\partial x} = x+y \longrightarrow f = \underbrace{\frac{1}{2}x^2 + xy + \cancel{h(y)}}_{\text{find } \partial f/\partial y}$ $-2y + C$

$\frac{\partial f}{\partial y} = \cancel{x} - 2 = 0 + \cancel{x} + h'(y) \rightarrow h'(y) = -2$

$h(y) = -2y$

$\boxed{f = \frac{x^2}{2} + xy - 2y + C}$ potential function

$f_x = x + y, \quad f_y = x - 2$

Problem 5. Given $\mathbf{F}(x,y) = \langle 2xy^3, \overset{Q}{3x^2y^2} \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by $\mathbf{r}(t) = \langle t^3 + 2t^2 - t, 3t^4 - t^2 \rangle$, $0 \le t \le 2$.

$\underbrace{f(\vec{r}(2)) - f(\vec{r}(0))}$

$Q_x = 6xy^2$

$P_y = 6xy^2$

$\langle f_x, f_y \rangle = \langle 2xy^3, 3x^2y^2 \rangle$

$\frac{\partial f}{\partial x} = 2xy^3 \rightarrow f = \underbrace{x^2y^3 + h(y)}_{\partial/\partial y}$

$\frac{\partial f}{\partial y} = 3\cancel{x^2}y^2 = 3\cancel{x^2}y^2 + h'(y) \rightarrow h(y) = C$

$f = x^2y^3 + C$

$f(14, 44) - f(0,0) = \underline{(14^2)(44^3) - 0}$

Problem 6. Let $\mathbf{F}(x,y) = \langle 3 + 2xy^2, 2x^2y \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the arc of the

~~parabola y = x^2 from (0, 0) to (1, 1).~~

hyperbola $y = \frac{1}{x}$ from $(1,1)$ to $\left(4, \frac{1}{4}\right)$.

$Q_x = 4xy \qquad P_y = 4xy$

$\frac{\partial f}{\partial x} = 3 + 2xy^2 \rightarrow f = \underbrace{3x + x^2y^2 + h(y)}_{\partial/\partial y}$ $= C$

$\frac{\partial f}{\partial y} = 2\cancel{x^2}y = 0 + 2\cancel{x^2}y + h'(y) = 0$

$P \quad 2y, x^2y^2$

$$\frac{\partial x}{}$$

$$f = 3x + x^2y^2$$

$$\frac{\partial f}{\partial y} = 2\cancel{x^2}y = 0 + 2\cancel{x^2}y + h'(y) = 0 \quad \partial/\partial y$$

$$f\left(4, \frac{1}{4}\right) - f(1,1) = 12 + 16\frac{1}{16} - (3+1) =$$

$$= 12 + 1 - 4 = \underline{\underline{9}}$$

Problem 7. Given $\mathbf{F}(x, y) = \langle \overset{P}{3x^2 - 4y}, \overset{Q}{4y^2 - 2x} \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by $\mathbf{r}(t) = \langle t^2, t^2 + t - 2 \rangle$, $0 \leq t \leq 1$.

$$Q_x - P_y = -2 - (-4) \neq 0$$

$$(t^2 + t - 2)(t^2 + t - 2)$$

$$\langle 3t^4 - 4(t^2 + t - 2), 4(t^2 + t - 2)^2 - 2t^2 \rangle \cdot \langle 2t, 2t + 1 \rangle$$

$$3t^4 - 4t^2 - 4t + 8, \quad 4$$

$\int$ polynomial

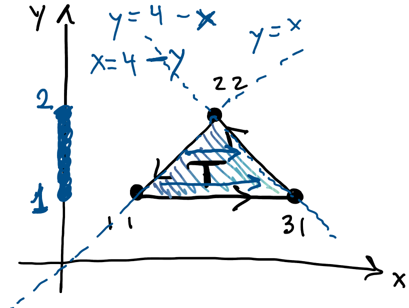
Green's Theorem: Let C be a positively oriented (counterclockwise) piecewise-smooth simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

This says that the line integral over a simple closed curve C is equal to a double integral over the area of the region D the curve C encloses.

Note: We only use Green's theorem if we are on a **positively oriented closed** curve. If the curve is not positively oriented, then change the sign of the line integral. If not explicitly stated, assume counterclockwise orientation.

Problem 8. Evaluate $\oint_C \overset{P}{y^2}\, dx + \overset{Q}{x}\, dy$ where C is the triangular path from $(1, 1)$ to $(3, 1)$ to $(2, 2)$ then back to $(1, 1)$.



$Q_x - P_y = 1 - 2y$

$\iint_T (1-2y)\, dA$

$\begin{cases} 1 \le y \le 2 \\ y \le x \le 4-y \end{cases}$

$\int_1^2 \int_y^{4-y} (1-2y)\, dx\, dy$

$(1-2y)[4-y-y] =$

$= (1-2y)(4-2y) =$

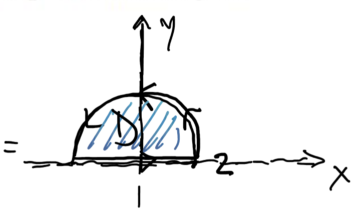
$= \int_1^2 4 - 2y - 8y + 4y^2 \; dy$

$(4 - 10y + 4y^2)\, dy$

$\left[4y - \cancel{\frac{10}{2}}^{5} y^2 + \frac{4}{3} y^3\right]_1^2 = 8 - 20 + \frac{32}{3} - \left(4 - 5 + \frac{4}{3}\right) = -11 + \frac{28}{3} =$

$= -\frac{5}{3}$

Problem 9. Evaluate $\oint_C \overset{P}{y^2}\, dx + \overset{Q}{x^2}\, dy$ where C is the boundary of the region bounded by the semicircle $y = \sqrt{4 - x^2}$ and the x axis. Assume positive orientation.



$\iint_D 2(x-y)\, dA = \int_0^{\pi} \int_0^{2} 2r(\cos\theta - \sin\theta)\, r\, dr\, d\theta =$

$= 2\left[\sin\theta + \cos\theta\right]_0^{\pi} \left[\frac{r^3}{3}\right]_0^2 =$

$= 2[-1 - 1]\left[\frac{8}{3} - 0\right] = -4\left(\frac{8}{3}\right) = -\frac{32}{3}$

$$= 2\,[-1-1]\left[\frac{8}{3}-0\right] = -4\left(\frac{8}{3}\right) = -\frac{32}{3}$$

Problem 10. Suppose a particle travels one revolution clockwise around the unit circle under the force field $\mathbf{F}(x,y) = \langle \underbrace{e^x - y^3}_{P}, \underbrace{\cos(y) + x^3}_{Q} \rangle$. Find the work done.

$$Q_x - P_y = 3x^2 - (-3y^2) = 3(x^2+y^2)$$

$$-\iint_D 3(x^2+y^2)\,dA = -\int_0^{2\pi}\int_0^1 3r^2\, r\,dr\,d\theta$$

$$-2\pi \left.\frac{3}{4}r^4\right|_0^1 = -2\pi\,\frac{3}{4} = -\frac{6}{4}\pi = -\frac{3}{2}\pi$$

Section 16.5

Definition: The **del operator**, denoted by ∇, is defined as $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

Definition of curl and divergence:

Problem 11. Find the divergence and curl of $\mathbf{F} = \langle \overset{P}{xy}, \overset{Q}{xz}, \overset{R}{xyz^2} \rangle$.

$$\operatorname{div}(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = y + 0 + 2xyz = y + 2xyz.$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & xyz^2 \end{vmatrix} = (xz^2 - x)\hat{i} - (yz^2 - 0)\hat{j} + (z - x)\hat{k}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & e^y \sin z & e^y \cos z \end{vmatrix} = (e^y \cos z - e^y \cos z)\,\hat{i} - (0-0)\,\hat{j} + (0-0)\,\hat{k} =$$

$$= \vec{0}$$

Theorem: If **F** is a vector field defined on all of $\Re^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then **F** is a conservative vector field. This gives us a way to determine whether a vector function on $\Re^3$ is conservative.

Problem 12. If $\mathbf{F} = \langle x,\ e^y \sin z,\ e^y \cos z\rangle$, Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{r}(t) = \langle t^4, t, 2t^2\rangle$, for $1 \le t \le 2$.

a) check F conserv. (curl $\vec{F} = \vec{0}$)

b) find pot. fnct. f $\quad \vec{\nabla} f = \vec{F}$

c) integral.

$$\frac{\partial f}{\partial x} = x \to \boxed{f = \underbrace{\frac{x^2}{2} + \cancel{g(y,z)}}_{\partial/\partial y} \quad e^y \sin z + h(z)} \ \partial/\partial z$$

$$\frac{\partial f}{\partial y} = e^y \sin z = 0 + \frac{\partial g}{\partial y} \to g = e^y \sin z + h(z)$$

$$\frac{\partial f}{\partial z} = \cancel{e^y \cos z} = 0 + \cancel{e^y \cos z} + \underbrace{h'(z)}_{\to 0} \quad h(z) = C$$

$$f = \frac{x^2}{2} + e^y \sin z + C \quad \text{Potential function}$$

$$f(\vec{r}(2)) - f(\vec{r}(1)) = f(16, 2, 8) - f(1,1,2) =$$

$$= \frac{16^2}{2} + e^2 \sin 8 - \left(\frac{1}{2} + e^1 \sin 2\right)$$