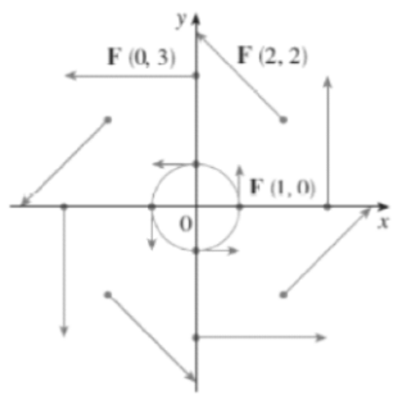
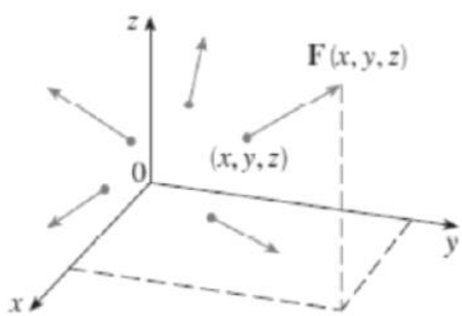
Wir 10: Sections 16.1, 16.2, 16.3, 16.4, 16.5

Section 16.1

Definition: A **vector field** in two dimension is a function $\mathbf{F}$ that assigns to each point (x, y) in $D \subset \mathbb{R}^2$ a two dimensional vector, $\mathbf{F}(x, y)$.
In two dimension, the vector field lies entirely in the xy plane.
Here is a vector field in $\mathbb{R}^2$:



Definition: A **vector field** in three dimension is a function $\mathbf{F}$ that assigns to each point (x, y, z) in $D \subset \mathbb{R}^3$ a three dimensional vector, $\mathbf{F}(x, y, z)$.
In three dimension, the vector field is in space.
Here is a vector field in $\mathbb{R}^3$:

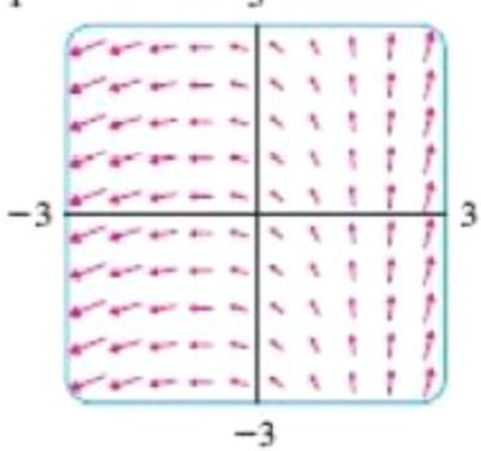


In order to match $\mathbf{F}$ with its vector field, choose a several points, (x, y), in each quadrant, and look at the *direction* of $\mathbf{F}(x, y)$. To narrow down further, look at the behavior of the components. Often times, it is a process of elimination.

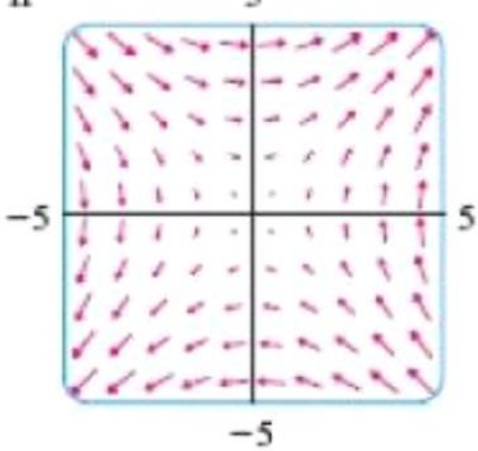
Problem 1. Match each vector field equation with its graph:

a) $\boldsymbol{F}(x, y) =< y, x >$
b) $\boldsymbol{F}(x, y) =< 1, \sin y >$
c) $\boldsymbol{F}(x, y) =< x - 2, x + 1 >$
d) $\boldsymbol{F}(x, y) =< y, \frac{1}{x} >$

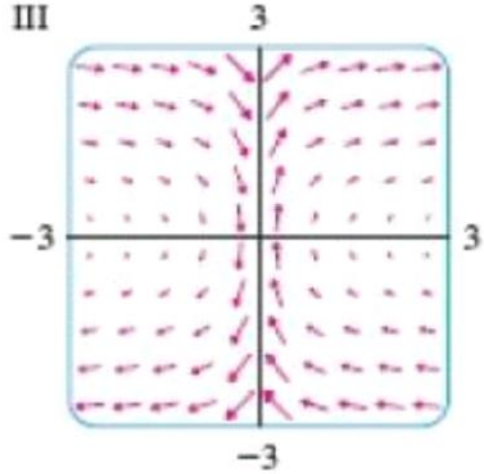
I



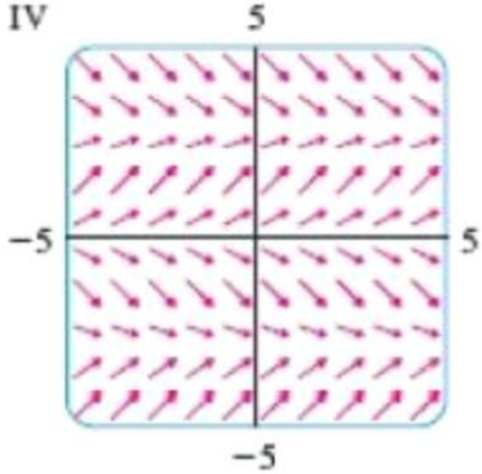
II



III



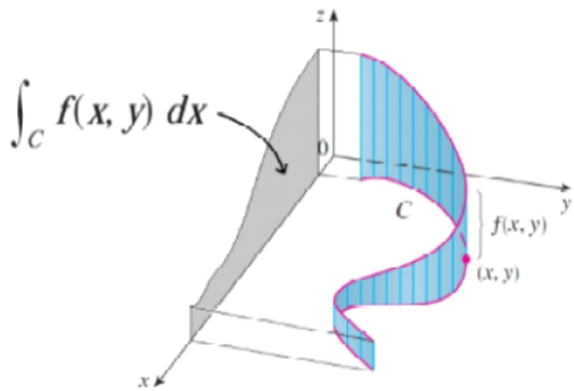
IV



Section 16.2

Definition: If f is defined on a smooth curve C defined as $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$, then the **line integral of f along C** is

$$\int_C f(x,y)ds = \int_a^b (f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b (f(x(t),y(t))|\mathbf{r}'(t)|\,dt$$



In order to find a line integral along a curce C, we must first parameteterize the curve. Sometimes, the parameterization will be given explicitly, other times you must parameterize the curve.

Problem 2. Evaluate $\int_C (2x+y)ds$, where C is defined as $\mathbf{r}(t) = \langle 2+t, 3-t\rangle$, $0 \le t \le 1$.

Problem 3. Set up but do not evaluate $\int_C (2x + x^2y)ds$, where C is the arc of the curve $y = x^2$ from $(1,1)$ to $(2,4)$ **using two different parameterizations.**

Problem 4. Evaluate $\int_C (x^2 + y)\ ds$ where C consists of the line segment from the point $(1,4)$ to $(3,-1)$.

Problem 5. Evaluate $\int_C (x+y)ds$, where C is the top half of the circle $x^2 + y^2 = 4$, oriented counterclockwise.

Problem 6. Set up but do not evaluate $\int_C (2 + x^2 y)ds$, where C is the arc of the curve $x = y^2$ from $(1, -1)$ to $(4, 2)$ and then along the line segment from the point $(4, 2)$ to the point $(3, 7)$.

Line Integrals over vector fields: Suppose now are moving a particle along a curve C through a vector (force) field, $\mathbf{F}$. We define the **line integral of F along C** to be

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

Problem 7. Find $\int_c \mathbf{F} \cdot \mathbf{r}$, where C is defined by $\mathbf{r}(\mathbf{t}) = \langle t, t^2, t^4 \rangle$, $0 \leq t \leq 1$, and $\mathbf{F}(x, y, z) = \langle x, z^2, -4y \rangle$.

Problem 8. Find the work done by the force field $\mathbf{F}(x, y) = \langle x^2, xy\rangle$ in moving an object counterclockwise around the right half of the circle $x^2 + y^2 = 9$.

Definition: Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ for $a \le t \le b$. The **line integral of f along** C **with respect to** x is $\int_C f(x, y)dx = \int_a^b (f(x(t), y(t))\ x'(t)\ dt$.

The **line integral of f along** C **with respect to** y is $\int_C f(x, y)dy = \int_a^b (f(x(t), y(t))\ y'(t)\ dt$

Problem 9. $\int_C ydx + x^2dy$, where C is decribed by $\mathbf{r}(t) = \langle 3e^t, e^{2t}\rangle$, $0 \le t \le 1$.

Problem 10. Evaluate $\int_C x dx + y dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.

Problem 11. Evaluate $\int_C (x+y)dz + (y-x)dy + zdx$ where C is decribed by $x = t^4$, $y = t^3$, $z = t^2$, $0 \leq t \leq 1$.

Section 16.3

In section 16.2, we learned how to find a line integral over a vector field $\mathbf{F}$ along a curve C that is parametrized by $\mathbf{r}(t)$, $a \leq t \leq b$.

Problem 1. Suppose we are moving a particle from the point $(0,0)$ to the point $(2,4)$ in a force field $\mathbf{F}(x,y) = \langle y^2, x\rangle$. Find $\int_c \mathbf{F} \cdot d\mathbf{r}$,where

a.) The particle travels along the line segment from $(0,0)$ to $(2,4)$.

b.) The particle travels along the curve $y = x^2$ from $(0,0)$ to $(2,4)$.

Note: Although the end points are the same, the value of the line integral is **different** because the **paths** are different. In this section, we will learn under what conditions the line integral is independent of the path taken.

Definition: If $\mathbf{F}$ is a continuous vector field, we say that $\int_c \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if and only if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 with the same starting and ending points. In other words, the line integral is the same **no matter what path** you travel on as long as the endpoints are the same.

Definition: A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar funtion f, that is there exists a function f so that $\mathbf{F} = \nabla f$. We call f the **potential function**.

Problem 2. Consider $f(x, y) = x^2y - y^3$. Find the gradient and explain why it is conservative. What is the potential function?

Recall the Fundamental Theorem of Calculus tells us that $\int_a^b f'(x)dx = f(b) - f(a)$.

Since $\nabla f = \langle f_x, f_y \rangle$, we can think of the potential function, f, as some sort of antiderivative of ∇f. Hence $\int \mathbf{F} \cdot d\mathbf{r} = \int \nabla f \cdot d\mathbf{r}$.

Fundamental Theorem for Line Integrals: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let $\mathbf{F}$ be a conservative vector field. Let f be a differentiable function of two or three variables whose gradient vector, ∇f, is continuous on C. Then

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Note: **Line integrals of conservative vectors fields are independent of path** because in a conservative vector field, the line integral is computed by only using the **endpoints** of the domain! Therefore, if we are in a conservative vector field, the line integral along a curve C in that vector field will be the same **no matter what curve we travel across** that connects the endpoints together. **WHICH MEANS WE DON'T EVEN NEED TO PARAMETERIZE THE CURVE!**

Question: How do we determine if a vector field is conservative, and if so, how do we find the potential function? The 'test for conservative' we use depends on whether $\mathbf{F}$ is in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Theorem: $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = P\mathbf{i} + Q\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, if and only if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

Note: This above criteria to determine if a vector field is conservative works only for $\mathbb{R}^2$.

Problem 3. Is $\mathbf{F}(x, y) = \langle 3x^2 - 4y, 4y^2 - 2x\rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

Problem 4. Is $\mathbf{F}(x, y) = \langle x + y, x - 2\rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

Problem 5. Given $\mathbf{F}(x,y) = \langle 2xy^3, 3x^2y^2 \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by $\mathbf{r}(t) = \langle t^3 + 2t^2 - t, 3t^4 - t^2 \rangle$, $0 \le t \le 2$.

Problem 6. Let $\mathbf{F}(x,y) = \langle 3 + 2xy^2, 2x^2y \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the arc of the

parabola y = x^2 from (0, 0) to (1, 1).

Problem 7. Given $\mathbf{F}(x, y) = \langle 3x^2 - 4y, 4y^2 - 2x \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by $\mathbf{r}(t) = \langle t^2, t^2 + t - 2 \rangle$, $0 \leq t \leq 1$.

Section 16.4

Green's Theorem: Let C be a positively oriented (counterclockwise) piecewise-smooth simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \, dA$$

This says that the line integral over a simple closed curve C is equal to a double integral over the area of the region D the curve C encloses.

Note: We only use Green's theorem if we are on a **positively oriented closed** curve. If the curve is not positively oriented, then change the sign of the line integral. If not explicitly stated, assume counterclockwise orientation.

Problem 8. Evaluate $\oint_C y^2\,dx + x\,dy$ where C is the triangular path from $(1,1)$ to $(3,1)$ to $(2,2)$ then back to $(1,1)$.

Problem 9. Evaluate $\oint_C y^2\,dx + x^2\,dy$ where C is the boundary of the region bounded by the semicircle $y = \sqrt{4 - x^2}$ and the x axis. Assume positive orientation.

Problem 10. Suppose a particle travels one revolution clockwise around the unit circle under the force field $\mathbf{F}(x, y) = \langle e^x - y^3, \cos(y) + x^3 \rangle$. Find the work done.

Section 16.5

Definition: The **del operator**, denoted by ∇, is defined as $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

Definition of curl and divergence:

Problem 11. Find the divergence and curl of $\mathbf{F} = \langle xy, xz, xyz^2 \rangle$.

Theorem: If $\mathbf{F}$ is a vector field defined on all of $\Re^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field. This gives us a way to determine whether a vector function on $\Re^3$ is conservative.

Problem 12. If $\mathbf{F} = \langle x,\ e^y \sin z,\ e^y \cos z \rangle$, Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{r}(t) = \langle t^4, t, 2t^2 \rangle$, for $1 \le t \le 2$.