MATHEMATICS
TEXAS A&M UNIVERSITY

Math Learning Center

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Math 152 – Fall 2022
"Week-in-Review"

1.

Determine whether the following series converges or diverges: $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$.

The series is alternating

i. $\frac{1}{\ln(n+1)} < \frac{1}{\ln(n)}$

ii. $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$

So the series converges by the AST

2.

Determine whether the following series converges or diverges: $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{\sqrt{n+1}}$.

The series is alternating.

However, $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1$

So the series diverges by the Divergence Test.

3.

Determine whether the following series converges or diverges: $\sum_{n=2}^{\infty} \frac{(-1)^n}{3n-1}$.

Alternating Series

i. $\frac{1}{3(n+1)-1} = \frac{1}{3n+2} < \frac{1}{3n-1}$ so $b_n \searrow$

ii. $\lim_{n} \frac{1}{3n-1} = 0$

The series converges by the AST

4.

Determine whether the following series converges or diverges: $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^3+1}$.

Alternating ✓

i. $\frac{n+1}{(n+1)^3+1} \overset{?}{<} \frac{n}{n^3+1}$

$\underbrace{(n^2+2n+1)(n+1)}_{(n+1)^3}$

$(n+1)(n^3+1) \overset{?}{<} n(n+1)^3 + n(1)$

$n^4+n+n^3+1 \overset{?}{<} n(n^3+n^2+2n^2+2n+n+1)+n$

$n^4+n^3+n+1 \overset{\text{yes!}}{<} n^4+3n^3+3n+2n$

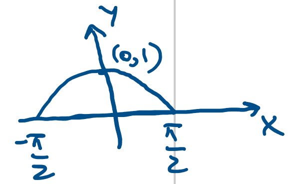
you can also do $f'(x)<0$ where $f(x) = \frac{x}{x^3+1}$

ii. $\lim_{n} \frac{n}{n^3+1} = \lim_{n} \frac{n}{n^3} = \lim_{n} \frac{1}{n^2} = 0$

series converges by AST.

5.

Determine whether the following series converges or diverges: $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\left(\frac{1}{n}\right)}{n^2}$.



$\frac{1}{n}$ is small, so $\cos\left(\frac{1}{n}\right) > 0$

$$0 < \cos\left(\tfrac{1}{n}\right) < 1$$

$$0 < \frac{\cos\left(\frac{1}{n}\right)}{n^2} < \frac{1}{n^2}$$

$$\sum_1^{\infty} \frac{\cos\left(\frac{1}{n}\right)}{n^2} \le \sum_1^{\infty} \frac{1}{n^2} \text{ convergent}$$

The given series converges by direct comparison to $\sum \frac{1}{n^2}$.

6.

Determine whether the following series converges or diverges: $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n n!}{(2n+1)!}$

Factorials & powers $\Rightarrow$ do Ratio Test.

$$\lim_n \left|\frac{a_{n+1}}{a_n}\right| = \lim_n \left| \frac{10^{\cancel{n+1}}(n+1)\cancel{!}}{(2n+3)\cancel{!}(2n+2)\cancel{10^n}} \cdot \frac{\cancel{(2n+1)!}}{\cancel{n!}} \right| =$$

$$= \lim_n 10 \frac{n+1}{(2n+3)(2n+2)} = 10 \lim_n \frac{n}{4n^2} = 0$$

The series is Absolutely Convergent hence convergent.

7.

(a)

Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$. Use the first 5 terms to estimate the sum. Estimate the error in the approximation s_5 to the sum of the series. (b) How many terms do you need to take in order to ensure an approximation to within .01?

$$s_5 = -1 + \frac{1}{2^5} - \frac{1}{3^5} + \frac{1}{4^5} - \frac{1}{5^5} = \ldots$$

a)

$|R_5| \le b_6 = \frac{1}{6^5}$ Answer to (a)

b)

Want $|R_n| < .01 = 10^{-2} = \frac{1}{100}$

make $b_{n+1} < \frac{1}{100}$

$$\frac{1}{(n+1)^5} < \frac{1}{100}$$

$$(n+1)^5 > 100$$

$$n+1 > \sqrt[5]{100}$$

$$n > \left(\sqrt[5]{100}\right) - 1$$ Answer to (b)

8.

State the Limit Comparison Test.

If Σa_n and Σb_n are two series with positive terms and $\lim_n \frac{a_n}{b_n} = c > 0$ and a finite number, then the two series are either both conv. or both div.

9.

Determine whether $\sum_{k=1}^{\infty} \frac{1}{k+1}$ converges or diverges.

$\sum \frac{1}{k+1} \le \sum \frac{1}{k}$ so direct comparison is inconclusive.

However, $\lim_{k} \frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k} \frac{k}{k+1} = 1$ so

both series diverge because the Harmonic series $\sum_{1}^{\infty} \frac{1}{n}$ diverges.

10.

Determine whether $\sum_{k=1}^{\infty} \frac{k+\sin k e^{-k}}{\sqrt{k^6-k^2}}$ converges or diverges.

Well, $(\sin k)e^{-k}$ is very tiny. For sure, it is

$-1 \le (\sin k)e^{-k} \le 1$ Add k

$k-1 \le k+(\sin k)e^{-k} \le k+1$

So the given series has POSITIVE TERMS

and $\frac{k+\sin k\, e^{-k}}{\sqrt{k^6-k^2}} < \frac{k+1}{\sqrt{k^6-k^2}}$ so if $\sum \frac{k+1}{\sqrt{k^6-k^2}}$ converges, so will the given series.

L.C.T. $\lim_{k} \frac{\frac{k+1}{\sqrt{k^6-k^2}}}{\frac{1}{k^2}} = \lim_{k} \frac{k^2(k+1)}{\sqrt{k^6-k^2}} = \lim_{k} \frac{k^3}{\sqrt{k^6}} = 1$

So both the given series and $\sum \frac{1}{k^2}$ converge.

11.

Determine whether $\sum_{k=1}^{\infty} \frac{k-\cos k}{k^2 \ln(k)^2 - k}$ converges or diverges.

↳ series has positive terms

$k-1 \le k - \cos k \le k+1$

It will conv/div. if $\sum \frac{k+1}{k^2(\ln k)^2 - k}$ does.

Note that $\sum \frac{k}{k^2(\ln k)^2} = \sum \frac{1}{k(\ln k)^2}$ diverges b/c.

$\int_1^\infty \frac{1}{x(\ln x)^2} dx = \int_0^\infty \frac{1}{u^2} du$ $(u = \ln x \ldots)$

$= -\frac{1}{u}\Big|_0^\infty = -\left(\frac{1}{\infty} - \frac{1}{0}\right) = \infty$

$\lim_k \frac{\frac{k+1}{k^2(\ln k)^2 - k}}{\frac{1}{k(\ln k)^2}} =$

$= \lim_k \frac{k(\ln k)^2(k+1)}{k^2(\ln k)^2 - k} = \lim_k \frac{k^2(\ln k)^2}{k^2(\ln k)^2}$

SO BOTH SERIES DIVERGE

12.

Determine whether $\sum_{k=1}^{\infty} \frac{k+1}{k^3-1}$ converges or diverges.

$k^3 - 1 = (k+1)(k^2 - k + 1)$ $\quad k^3 \cancel{-k^2} \cancel{+k} + \cancel{k^2} \cancel{-k} + 1$

$\sum \frac{\cancel{k+1}}{(\cancel{k+1})(k^2-k+1)}$ limit-compare with $\sum \frac{1}{k^2}$ which is convergent.

$\lim_k \frac{\frac{k+1}{k^3-1}}{\frac{1}{k^2}} = \lim_k \frac{k^2(k+1)}{k^3-1} = \lim_k \frac{k^3}{k^3} = 1,$

so both series converge.

13.

Determine whether $\sum_{k=1}^{\infty} \sin \frac{1}{k}$ converges or diverges.

When k is large, $\frac{1}{k}$ is very tiny, so $\sin\left(\frac{1}{k}\right) \approx \frac{1}{k}$ and $\sum \sin\frac{1}{k} \approx \sum \frac{1}{k}$, which diverges (Harmonic Series).

14.

What is an alternating series? What does the Alternating Series Test say? What is the remainder theorem for Alternating Series?

$\sum (-1)^n b_n \quad b_n > 0$

AST: If series is alternating and b_n is decreasing and $b_n \to 0$ as $n \to \infty$, then $\sum (-1)^n b_n$ converges.

15.

Alternating Harmonic Series.

Determine whether $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges or diverges. If it converges, how many terms need to be taken so that the Nth partial sum is within .1 of the sum?

AST alternating series ✓

i. $\frac{1}{k+1} < \frac{1}{k}$

ii. $\frac{1}{k} \to 0$ as $k \to \infty$

Series converges by AST

16.

Determine whether $\sum_{k=1}^{\infty} \frac{\cos \pi k}{\ln \ln k}$ converges or diverges. If it converges, how many terms need to be taken so that the Nth partial sum is within .1 of the sum?

$\cos(\pi k) = (-1)^k \qquad k = 0, 1, 2, \ldots$

$\sum \frac{(-1)^k k}{\ln(\ln k)}$

Note: $\lim_{k} \frac{k}{\ln(\ln k)} = \frac{\infty}{\infty} \overset{L'HR.}{=} \lim_{k\to\infty} \frac{1}{\frac{1}{\ln k} \cdot \frac{1}{k}} =$

$= \lim_{k} k \ln k = \infty$

The series diverges by the Divergence Test.

17.

Determine whether $\sum_{k=1}^{\infty} \frac{k(-1)^k}{k+1}$ converges or diverges. If it converges, how many terms need to be taken so that the Nth partial sum is within .1 of the sum?

Note: $\lim_{k\to\infty} \frac{k}{k+1} = 1 \neq 0$

The series diverge by the Div. Test.

18.

Determine whether $\sum_{k=1}^{\infty}(-1)^k \sin\frac{1}{k}$ converges or diverges. If it converges, how many terms need to be taken so that the Nth partial sum is within .1 of the sum?

Alternating, $\sin\frac{1}{k} \approx \frac{1}{k}$ which decreases and it approaches zero as $k \to \infty$.

So the series converges by the AST.

If we want $|R_n| < 0.1$ then make $b_{n+1} < \frac{1}{10}$, or

$\sin\left(\frac{1}{n+1}\right) < \frac{1}{10}$ or $\frac{1}{n+1} < \arcsin\left(\frac{1}{10}\right)$

or $n+1 < \frac{1}{\arcsin\frac{1}{10}}$.

$n < \frac{1}{\arcsin\frac{1}{10}} - 1$

$\arcsin \frac{1}{10}$

$$n < \frac{1}{\arcsin \frac{1}{10}} - 1$$

19.

Determine whether $\sum_{k=1}^{\infty}(-1)^k(\sqrt{k^2+1} - \underbrace{\sqrt{k^2}}_{=k})$ converges or diverges. If it converges, how many terms need to be taken so that the Nth partial sum is within .1 of the sum?

$$-(\sqrt{2}-\sqrt{1})+(\sqrt{5}-\sqrt{4})-(\sqrt{10}-\sqrt{9})+(\sqrt{17}-4)-(\sqrt{26}-5)\cdots$$

$$= -\sqrt{2}+1+\sqrt{5}-2-\sqrt{10}+3+\sqrt{17}-4-\sqrt{26}+5 \cdots$$

$$(-1)^k\left(\sqrt{k^2+1}-\sqrt{k^2}\right)\frac{\left(\sqrt{k^2+1}+\sqrt{k^2}\right)}{\sqrt{k^2+1}+\sqrt{k^2}} =$$

$$= (-1)^k\frac{k^2+1-k^2}{\sqrt{k^2+1}+\sqrt{k^2}} = \frac{(-1)^k}{\sqrt{k^2+1}+\sqrt{k^2}}$$

the series is convergent by the AST

$b_{n+1} < \frac{1}{10}$ or $\frac{1}{\sqrt{(n+1)^2+1}+\sqrt{(n+1)^2}} < \frac{1}{10}$

$\sqrt{(n+1)^2+1} + n+1 > 10$ n=5 will do

20.

Determine whether $\sum_{k=1}^{\infty}(-1)^k\frac{1}{k^2\cos k}$ converges or diverges. If it converges, how many terms need to be taken so that the Nth partial sum is within .1 of the sum?

The $\lim_{k\to\infty} \frac{1}{k^2 \cos k}$ does not exist.

The series diverges by the Test for Divergence.

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