Part I: Improper Integrals:

Problem 1. Compute $\int_{x=3}^{\infty} \frac{dx}{(x-2)^{3/2}}$.

$$\lim_{L \to +\infty} \int_3^L \frac{dx}{(x-2)^{3/2}}$$

$$\int (x-2)^{-3/2}\, dx = \int u^{-3/2}\, du = \quad \left[\begin{array}{l} u = x-2 \\ du = dx \end{array} \right.$$

$$= \frac{1}{-\frac{1}{2}} u^{-3/2 + \frac{2}{2}} = -2\, u^{-1/2} =$$

$$= -2\,(x-2)^{-1/2} \Big|_3^L =$$

$$= -2\left[(L-2)^{-1/2} - (3-2)^{-1/2}\right] =$$

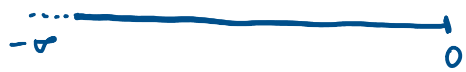
$$= -2\left[\frac{1}{\sqrt{L-2}} - \frac{1}{\sqrt{1}}\right] \text{ when } L \to +\infty$$

$$\frac{1}{\sqrt{L-2}} \to 0$$

Answer: $\int_3^{\infty} \frac{dx}{(x-2)^{3/2}}$ convergesto 2.

Problem 2. Compute $\int_{\underbrace{x=-\infty}}^{0} \frac{dx}{3-4x}$.

$$\lim_{N \to -\infty} \int_{N}^{0} \frac{dx}{3-4x}$$



$$\int \frac{dx}{3-4x} = \int -\frac{1}{4}\frac{1}{u}\,du =$$

$u = 3-4x$

$du = -4\,dx$

$dx = -\frac{du}{4}$

$$= -\frac{1}{4}\ln|u| = -\frac{1}{4}\ln|3-4x| \Big|_{N}^{0} =$$

$$= -\frac{1}{4}\left[\ln|3-0| - \underbrace{\ln|3-4N|}\right] \text{ when } N \to -\infty$$

$$-\frac{1}{4}\left[\ln 3 \quad - \infty\right] \to +\infty$$

$\int_{-\infty}^{0} \frac{dx}{3-4x}$ is divergent to $+\infty$

Problem 3. Compute $\int_{x=0}^{\infty} te^{-t}\,dt$

$$\lim_{L\to+\infty}\int_0^L t e^{-t}\,dt$$

$$uv-\int v\,du = \int u\,dv$$

$u=t$ $\quad e^{-t}dt = dv$

$du=dt$ $\quad v=-e^{-t}$

$$\int t e^{-t}\,dt =$$

$$= -te^{-t} - \int -e^{-t}\,dt = -te^{-t} - e^{-t}\Big|_0^L =$$

$$= \left(\underbrace{-Le^{-L}}_{0} - \underbrace{e^{-L}}_{0}\right) - \left(-0 - e^{0}\right)$$

when $L \to +\infty$

$\lim_{x\to\infty} -xe^{-x}$ $\quad -\infty\cdot 0$ IND. FORM

$\lim_{x\to\infty} -\frac{x}{e^{x}}$ $\quad \frac{\infty}{\infty}$ L'HR.

$\lim_{x\to\infty} -\frac{1}{e^{x}} = 0$

$\int_0^{\infty} te^{-t}\,dt$ converges to 1.

Problem 4. Compute $\int_{x=2}^{\infty} \frac{dx}{x^2+2x-3}$.

$x^2+2x-3=(x+3)(x-1)$

$$\lim_{L\to\infty}\int_2^L \frac{dx}{(x+3)(x-1)} =$$

$$\frac{1}{(x+3)(x-1)} = \frac{A}{x+3}(x-1) + \frac{B}{x-1}(x+3)$$

$$0x+1 = A(x-1)+B(x+3)$$

$$0 = A + B$$

$$1 = -A + 3B = B + 3B$$

$$1 = 4B \quad B = 1/4 \quad A = -1/4$$

$$= \lim_{L\to\infty}\int_2^L \left(\frac{-1/4}{x+3} + \frac{1/4}{x-1}\right)dx$$

$$-\frac{1}{4}\ln|x+3| + \frac{1}{4}\ln|x-1| \Big|_2^L$$

$$= \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| \Big|_2^L =$$

$$= \frac{1}{4}\left[\underbrace{\ln\left|\frac{L-1}{L+3}\right|} - \ln\left|\frac{2-1}{2+3}\right|\right] \to$$

$$\lim_{x\to\infty} \ln\left|\frac{x-1}{x+3}\right| =$$

$$= \ln\left(\lim_{x\to\infty}\frac{x-1}{x+3}\right)$$

$\frac{\infty}{\infty}$ HR $\frac{1}{1}$

$$\ln(1)$$

$$\to \frac{1}{4}\left[0 - \ln\frac{1}{5}\right]$$

$$\frac{1}{4}\left(\ln\left(\frac{1}{5}\right)^{-1}\right) =$$

$$= \frac{1}{4}\ln 5$$

the given integral converges to this.

Problem 5. What power does r need to be for the integral $\int_{t=1}^{\infty} \frac{dt}{t^r}$ to converge?

$$\int_1^\infty \frac{dt}{t^r} = \lim_{L \to \infty} \int_1^L \frac{dt}{t^r}$$

$$\int t^{-r}\,dt = \left.\frac{t^{-r+1}}{-r+1}\right|_1^L = \left(L^{1-r} - 1^{1-r}\right)\frac{1}{(1-r)}$$

$L^{1-r} \to \infty$ if $r<1$

?

$$\frac{1}{(L)^{r-1}} \text{ positive} \qquad \lim_{L \to \infty} L^{1-r}$$

if $1-r<0$ then the limit is zero

$$\boxed{1<r}$$

The $\int_1^\infty \frac{dx}{x^r}$ converges to $\frac{1}{r-1}$ when $r>1$

IF $r=1$ $\int_1^\infty \frac{dx}{x}$ is divergent

$$\ln|x|\Big|_1^L = \ln L - \underbrace{\ln 1}_{0} \to \infty \text{ as } L \to \infty$$

Problem 6. What power does r need to be for the integral $\int_{t=1}^{\infty} \frac{\ln t dt}{t^r}$ to converge?

$$\lim_{L\to\infty} \int_1^L \frac{\ln t}{t^r}\,dt$$

$$\int \frac{\ln t}{t^r}\,dt \qquad u = \ln t \qquad dv = t^{-r}\,dt$$

$$du = \frac{1}{t}dt \qquad v = \frac{t^{1-r}}{1-r}$$

$$(\ln t)\left(\frac{t^{1-r}}{1-r}\right) - \int \frac{1}{1-r}\,\underbrace{t^{-1}\,t^{1-r}}\,dt \qquad \cancel{1}\cancel{+1}-r = -r$$

$$(\ln t)\left(\frac{t^{1-r}}{1-r}\right) + \int \frac{1}{r-1}\,t^{-r}\,dt$$

$$\left[\frac{1}{1-r}(\ln t)t^{1-r} - \frac{1}{\underline{\underline{1-r}}}\,\frac{t^{1-r}}{\underline{\underline{1-r}}}\right]_1^L =$$

$$\left[\left(\frac{1}{1-r}\right)\underbrace{(\ln L)\,L^{1-r}} - \frac{1}{(1-r)^2}\underbrace{L^{1-r}}\right] - \left[0 - \frac{1}{(1-r)^2}\,1^{1-r}\right]$$

? As $L \to \infty$?

If $1-r<0$ then $L^{1-r} \to 0$ $\;\; = \frac{1}{L^{r-1}}$

$\boxed{1<r}$

IF $\boxed{r>1}$ $\quad \frac{\ln L}{L^{r-1}} \to \frac{\infty}{\infty}$ L'HR $\quad \frac{\frac{1}{L}}{(r-1)L^{r-2}} \cdot \frac{1}{(r-1)L^{r-1}} \to 0$

The given $\int$ converges to $\frac{1}{(1-r)^2}$ when $r>1$.

$(1-r)^2$

Problem 7. Does the integral $\int_{t=e}^{\infty} \frac{dt}{t(\ln t)^r}$ converge for any value of r?

$$\lim_{L\to\infty}\int_e^L \frac{dt}{t(\ln t)^r} \qquad u=\ln t \quad du=\frac{1}{t}\,dt$$

$$\int \frac{dt}{t(\ln t)^r} = \int \frac{du}{u^r} = \int u^{-r}\,du = \frac{u^{-r+1}}{-r+1} =$$

$$= \frac{(\ln t)^{1-r}}{1-r}\Bigg|_e^L = \frac{1}{1-r}\left[(\ln L)^{1-r} - \overset{-1}{(\ln e)^{1-r}}\right]$$

when $L\to\infty$, $(\ln L)^{1-r}\to 0$ if $1-r<0$, $1<r$ (otherwise it diverges)

The given integral converges to $\frac{-1}{1-r} = \frac{1}{r-1}$ $\quad \boxed{r>1}$

Problem 8. Determine whether the integral $\int_{t=2}^{\infty} \frac{dt}{t^2-1}$ converges.

$$\lim_{L\to\infty} \int_2^L \frac{dt}{t^2-1}$$

$$\frac{1}{(t+1)(t-1)} = \frac{A(t-1)}{(t+1)(t-1)} + \frac{B(t+1)}{(t-1)(t+1)}$$

$$0t + 1 = A(t-1) + B(t+1)$$

$$0 = A + B$$

$$1 = -A + B = B + B$$

$$B = \frac{1}{2} \quad A = -\frac{1}{2}$$

$$\frac{1}{2}\int \left(\frac{-1}{t+1} + \frac{1}{t-1}\right) dt$$

$$\frac{1}{2}\left[-\ln|t+1| + \ln|t-1|\right] =$$

(from 1 to 2 it diverges)

$$= \frac{1}{2} \ln\left|\frac{t-1}{t+1}\right| \Bigg|_2^L =$$

$$= \frac{1}{2}\left[\ln\left|\frac{L-1}{L+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right] =$$

$$= \frac{1}{2}\left(0 - \ln\frac{1}{3}\right)$$

$$\ln\left(\frac{1}{3}\right)^{-1}$$

$$\ln 3$$

$$\lim_{L\to\infty} \ln\left|\frac{L-1}{L+1}\right|$$

$$\ln\Bigg(\underbrace{\lim_{L\to\infty} \frac{L-1}{L+1}}_{1}\Bigg) = \ln 1 = 0$$

$$\int_2^{\infty} \frac{dt}{t^2-1} \text{ converges to } \frac{1}{2}\ln 3.$$

Problem 9. Determine whether the integral $\int_{t=1}^{\infty} \frac{dt}{t+1}$ converges.

$$\lim_{L\to\infty} \int_1^L \frac{dt}{t+1}$$

sol I

$$\int_1^\infty \frac{dt}{t+1} = \ln|t+1|\Big|_1^\infty = \ln|\infty+1| - \ln|1+1| =$$
$$= \infty - \ln 2 = \infty$$

$\int_1^\infty \frac{dt}{t+1}$ diverges to $+\infty$

sol II comparison test (bigger than a divergent)

$t+1 > t$ so $\int \frac{1}{1+t} < \int \frac{1}{t}$ ...

$\boxed{\frac{4}{1+t} \overset{?}{>} \frac{1}{t}}$ as long as $4t > 1+t$

$3t > 1$

$t > 1/3$

$\int \frac{1}{1+t} > \int \frac{4}{t}$ divergent.

Problem 10. Determine the values of r for which the integral $\int_{t=0}^{\infty} \frac{dt}{t^r(t+1)}$ converges.

$$\int_0^1 \frac{dt}{t^r(t+1)} + \int_1^\infty \frac{dt}{t^r(t+1)}$$

$$\boxed{\int_1^\infty \frac{dt}{t^r(t+1)} \quad \text{convergent when } r>0}$$

$$t^r(t+1) = t^{r+1} + t^r > t^{r+1}$$

$$\frac{1}{t^r(t+1)} < \frac{1}{t^{r+1}} \text{ conv?}$$

$$\int t^{-r-1}\,dt = \frac{t^{-r\cancel{-1}\cancel{+1}}}{-r}$$

if $-r<0$ conv

$r>0$

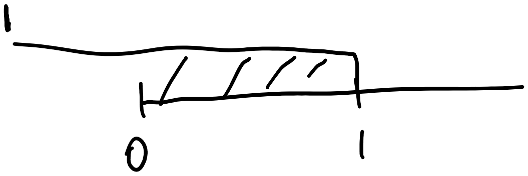
$$\boxed{\int_0^1 \frac{dt}{t^r(t+1)} \quad \text{convergent when } r<1}$$

at least 1

$$\overbrace{0<t}<1 \Rightarrow t^r\overbrace{(t+1)} \geq t^r \cdot 1 = t^r$$

$$\frac{1}{t^r(t+1)} \leq \frac{1}{t^r}$$

$\int_0^1 \frac{1}{t^r}\,dt$ conv when $r<1$



So $\int_0^\infty \frac{dt}{t^r(t+1)}$ converges when $0<r<1$

PART II : sequences.

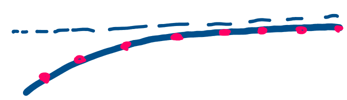
Problem 11. Determine whether the following sequence converges. If it does, find its limit: $a_k = \frac{3+5k^2}{k+k^2}$.

$$f(x) = \frac{3+5x^2}{x+x^2} \quad \frac{\infty}{\infty} \quad \mathcal{L}'H.R$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{0+10x}{1+2x} = \frac{\infty}{\infty}\ \mathcal{L}'HR =$$

$$= \lim_{x\to\infty} \frac{10}{2} = 5$$

$$\lim_{k} a_k = 5.$$



$$\frac{3+5k^2}{k+k^2} = \frac{\cancel{k^2}\left(\frac{3}{k^2}+5\right)}{\cancel{k^2}\left(\frac{k}{k^2}+1\right)} \longrightarrow \frac{5}{1} = 5$$

$$\frac{3}{k^2} \to 0 \qquad \frac{k}{k^2} \to \frac{1}{k} \to 0$$

Problem 12. Determine whether the following sequence converges. If it does, find its limit: $a_k = \frac{3+5k^2}{k+k^2}$.

done ✓

Problem 13. Determine whether the following sequence converges. If it does, find its limit: $a_n = \frac{3\sqrt{n}}{\sqrt{n}+2}$.

$$a = \frac{3\sqrt{n}}{\sqrt{n}+2} = \frac{\cancel{\sqrt{n}}(3)}{\cancel{\sqrt{n}}\left(1+\underbrace{\frac{2}{\sqrt{n}}}_{\to 0}\right)} \longrightarrow \frac{3}{1}$$

$$\lim_n a_n = 3$$

do L'HR as an exercise.

Problem 14. Determine whether the following sequence converges. If it does, find its limit: $a_k = \cos\frac{k\pi}{k+1}$.

$$f(x) = \cos\frac{\pi x}{x+1}$$

$$\lim_{x\to\infty} \cos\left(\frac{\pi x}{x+1}\right) = \cos\left(\lim_{x\to\infty}\frac{\pi x}{x+1}\right) =$$

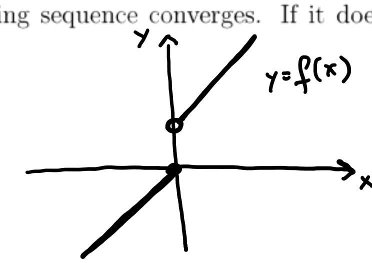
$$= \cos\pi = -1$$

$$\lim_{x\to\infty}\frac{\pi x}{x+1} \quad \frac{\infty}{\infty}$$

$$\overset{LHR}{=} \lim_{x\to\infty}\frac{\pi}{1}$$

$$\lim_{k} a_k = -1$$

Problem 15. Determine whether the following sequence converges. If it does, find its limit: $a_k = f(\frac{1}{k})$. Here, $f(x) = \begin{cases} x, & x \leq 0 \\ 1+x, & 0 < x \end{cases}$.



$a_1 = f(1) = 2 \qquad a_2 = f\left(\frac{1}{2}\right) = \frac{3}{2}$

$a_k = f\left(\frac{1}{k}\right) = 1 + \frac{1}{k} = \frac{k+1}{k}$

$a_k = \frac{k+1}{k} \qquad \frac{x+1}{x} \longrightarrow \frac{1}{1}$

$\lim_{k} a_k = 1$

Problem 16. Determine whether the following sequence converges. If it does, find its limit: $a_k = f(\frac{(-1)^k}{k})$. Here, $f(x) = \begin{cases} x, & x \le 0 \\ 1+x, & 0 < x \end{cases}$.

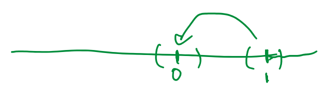
$a_1 = f\left(\frac{(-1)^1}{1}\right) = f(-1) = -1$ $\quad a_2 = f\left(\frac{(-1)^2}{2}\right) = f\left(\frac{1}{2}\right) = \frac{1}{2}+1 = \frac{3}{2}$

$a_3 = f\left(\frac{-1}{3}\right) = -\frac{1}{3}$ $\quad a_4 = f\left(\frac{1}{4}\right) = \frac{1}{4}+1 = \frac{5}{4}$

$a_5 = \frac{1}{5}+1$

$-1, \frac{3}{2}, -\frac{1}{3}, \frac{5}{4}, -\frac{1}{5}, \frac{6}{5}, -\frac{1}{7}, \frac{7}{6} \dots\dots$

$a_{2k+1} = \frac{-1}{2k+1} \to 0$



$a_{2k} = \frac{2k+1}{2k} \xrightarrow{k\to\infty} \frac{2}{2} = 1$

This a_k diverges, no limit.

Problem 17. Determine whether the following sequence converges. If it does, find its limit: $a_k = \ln(k+1) - \ln(k)$. $\infty - \infty$ ind. form

$$a_k = \ln \frac{k+1}{k}$$

$$\lim_k a_k = \lim_k \ln\left(\frac{k+1}{k}\right) =$$

$$= \ln\left(\underbrace{\lim_{k\to\infty} \frac{k+1}{k}}_{1}\right) = \ln 1 = 0 .$$

$$a_k \to 0$$

Problem 18. Determine whether the following sequence converges. If it does, find its limit: $a_k = \sqrt{k+1} - \sqrt{k}$. $\infty - \infty$

$$a_k = \left(\sqrt{k+1} - \sqrt{k}\right) \cdot \frac{\left(\sqrt{k+1} + \sqrt{k}\right)}{\sqrt{k+1} + \sqrt{k}} =$$

$$= \frac{\left(\sqrt{k+1}\right)^2 - \left(\sqrt{k}\right)^2}{\sqrt{k+1} + \sqrt{k}} = \frac{\cancel{k} + 1 - \cancel{k}}{\sqrt{k+1} + \sqrt{k}} = \frac{1}{\sqrt{k+1} + \sqrt{k}}$$

$k \to \infty$ $\lim_{k \to \infty}$ is $\frac{1}{\infty + \infty} = \frac{1}{\infty} = 0$

Problem 19. Determine whether the following sequence converges. If it does, find its limit: $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\}$.

$a_{n+1} = \sqrt{2a_n}$

$\sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}$

$a_n \nearrow$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad \cdots \quad a_{n-1} \quad a_n \quad a_{n+1}$

$a_n \to \ell \qquad a_{n+1} \longrightarrow \ell$

$$a_{n+1} = \sqrt{2a_n}$$

$$\ell = \sqrt{2\ell}$$

$$\ell^2 = 2\ell$$

$$\ell^2 - 2\ell = 0$$

$$\ell(\ell - 2) = 0$$

$\cancel{\ell = 0}$

$\ell = 2$

$\sqrt{n} \to \infty$

$\sqrt{n+1} \to \infty$

$\frac{1}{n} \to 0$

$\frac{1}{n+1} \to 0$

Problem 20. Determine whether the following sequence converges. If it does, find its limit: $a_1 = 1$ and $a_k = 3 - \frac{1}{a_{k-1}}$.

$a_1 = 1$ $\quad a_2 = 3 - \frac{1}{1} = 2$ $\quad a_3 = 3 - \frac{1}{2} = \frac{5}{2}$

$a_4 = 3 - \frac{2}{5} = \frac{13}{5}$ 2.6 $\quad a_5 = 3 - \frac{5}{13} = \frac{34}{13} = \underline{2.61538}$

$a_k \nearrow$

If $a_k \to l$, so does $a_{k-1} \to l$

$$l = 3 - \frac{1}{l} \Rightarrow l^2 = 3l - 1$$

$$l^2 - 3l + 1 = 0$$

$$l = \frac{3 \pm \sqrt{9-4}}{2} = $$

$$= \frac{3 \pm \sqrt{5}}{2}$$

$$\frac{3-\sqrt{5}}{2} = \underline{0.38196\ldots}$$

$$\boxed{l = \frac{3+\sqrt{5}}{2}}$$