Trigonometric Substitution & Partial Fractions

Problem 1. Compute $\int \frac{x^2}{\sqrt{9-x^2}}dx$.

$x = 3\sin\theta \qquad dx = 3\cos\theta \, d\theta$

$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) =$
$= 9\cos^2\theta$

$\underbrace{\sin\theta = \frac{x}{3}} \Rightarrow \theta = \arcsin\left(\frac{x}{3}\right)$

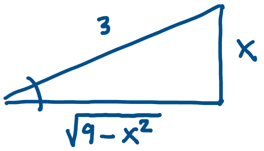
$$\int \frac{9\sin^2\theta}{\cancel{3\cos\theta}} \cancel{3\cos\theta} \, d\theta$$

$$\int 9 \, \frac{1}{2}(1 - \cos 2\theta)\, d\theta = \frac{9}{2}\left[\theta - \underbrace{\frac{1}{2}\sin 2\theta}\right] =$$

$= -\frac{1}{\cancel{2}} \cancel{2} \sin\theta\cos\theta =$

$= -\frac{x}{3}\,\frac{\sqrt{9-x^2}}{3}$

$$= \frac{9}{2}\left[\arcsin\frac{x}{3} - \frac{x\sqrt{9-x^2}}{9}\right] + C$$



Problem 2. Compute $\int_{\underline{x=0}}^{\underline{2}} \frac{x}{\sqrt{36-4x^2}}dx$.

$$\int_0^{\arcsin\frac{2}{3}} \frac{3\sin\theta}{2\cancel{6}\,\cancel{\cos\theta}}\,\cancel{3}\cancel{\cos\theta}\,d\theta =$$

$$= \frac{3}{2}\int_0^{\arcsin\frac{2}{3}} \sin\theta\,d\theta =$$

$$= \frac{3}{2}\left[-\cos\theta\right]_0^{\arcsin\frac{2}{3}} =$$

$$= -\frac{3}{2}\left[\cos\left(\arcsin\frac{2}{3}\right) - \cos 0\right]$$

$$-\frac{3}{2}\left[\frac{\sqrt{5}}{3} - 1\right] =$$

$$= \frac{3}{2} - \frac{\sqrt{5}}{2}.$$

$1 - \sin^2 = \cos^2$

$\underline{\underline{36}} - \underline{\underline{4}}x^2 = \underline{\underline{36}}\cos^2$

$4x^2 = 36\sin^2$

$x^2 = \frac{36}{4}\sin^2$

$x = \frac{6}{2}\sin\theta$

$x = 3\sin\theta \quad dx = 3\cos\theta\,d\theta$

$36 - 4x^2 = \underline{\underline{36}} - \underline{\underline{4}}(9\sin^2\theta) =$

$= 36(1 - \sin^2\theta) = \underline{36\cos^2\theta}$

$0 = \frac{6}{2}\sin\theta \Rightarrow \theta = 0$

$2 = \frac{6}{2}\sin\theta \Rightarrow \sin\theta = \frac{4}{6} = \frac{2}{3}$

$\theta = \arcsin\frac{2}{3}$

$\sin\theta = \frac{x}{3}$

$\sqrt{9-x^2} = \sqrt{9-4} = \sqrt{5}$

$\cos\theta = \frac{\sqrt{9-x^2}}{3} = \frac{\sqrt{5}}{3}$

Problem 3. Compute $\int_{t=0}^{2} \frac{1}{\sqrt{4+t^2}} dt$.

$t = 2\tan\theta \qquad dt = 2\sec^2\theta \, d\theta$

$\tan\theta = \frac{t}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{t}{2}\right)$

$4+t^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) =$

$= 4\sec^2\theta$

$$\int_{\tan^{-1}\left(\frac{0}{2}\right)}^{\tan^{-1}\left(\frac{2}{2}\right)} \frac{1}{\cancel{2}\cancel{\sec\theta}} \cancel{2}\sec^{\cancel{2}}\theta \, d\theta$$

$$\int_{0}^{\pi/4} \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| \Big|_{\theta=0}^{\pi/4} =$$

$$\ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0| =$$

$$= \ln\left|\frac{2}{\sqrt{2}} + 1\right| - \underbrace{\ln|1+0|}_{\text{zero}} =$$

$$= \ln|1+\sqrt{2}| .$$

Problem 4. Compute $\int_{t=0}^{a} t^2\sqrt{a^2-t^2}dt$.

$t = a\sin\theta \Rightarrow dt = a\cos\theta\, d\theta$

$\sin\theta = \frac{t}{a}$

$a^2-t^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) =$

$= a^2\cos^2\theta$

$t=0 \to \sin\theta = 0 \Rightarrow \theta = 0$

$t=a \to a = a\sin\theta \Rightarrow \sin\theta = 1 \to \theta = \pi/2$

$$\int a^2\sin^2\theta\, a\cos\theta\, a\cos\theta\, d\theta = \int a^4 \sin^2\theta\cos^2\theta\, d\theta \ *$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$

$\frac{1}{4}\sin^2 2\theta = \sin^2\theta\cos^2\theta$

$\frac{1}{4}\frac{1}{2}(1-\cos 4\theta)$

$\frac{1}{8} - \frac{1}{8}\cos 4\theta$

$\frac{1}{2}[1-\cos 2\theta]\frac{1}{2}[1+\cos 2\theta]$

$\frac{1}{4}[1-\cos^2 2\theta]$

$\frac{1}{4}[1-\frac{1}{2}(1+\cos 4\theta)]$

$\frac{1}{4} - \frac{1}{8}(1+\cos 4\theta)$

$\underbrace{\frac{1}{4} - \frac{1}{8}}_{} - \frac{1}{8}\cos 4\theta$

$\frac{1}{8} - \frac{1}{8}\cos 4\theta$

$$* \ a^4\int_0^{\pi/2}\left(\frac{1}{8} - \frac{1}{8}\cos 4\theta\right) d\theta =$$

$$= \frac{1}{8}\theta - \frac{1}{8}\frac{1}{4}\sin 4\theta \Big|_0^{\pi/2} =$$

$$a^4\left\{\frac{1}{8}\frac{\pi}{2}\underbrace{-\frac{1}{32}\sin 2\pi}_{} - (0-0)\right\} =$$

$$= a^4\frac{\pi}{16}.$$

Problem 5. Compute $\int \frac{1}{t^2+2t+5} dt$.

$\underbrace{t^2+2t+1}-1+5=(t+1)^2+4$

$t+1=2\tan\theta$

$\tan\theta=\frac{t+1}{2} \Rightarrow \theta=\tan^{-1}\left(\frac{t+1}{2}\right)$

$\int \frac{1}{(t+1)^2+4}\,dt$

$dt=2\sec^2\theta\,d\theta$

$(t+1)^2+4=4\tan^2\theta+4=$

$=4(\tan^2\theta+1)=4\sec^2\theta$

$\int \frac{1}{4\cancel{\sec^2\theta}}2\cancel{\sec^2\theta}\,d\theta$

$\int \frac{1}{2}d\theta=\frac{1}{2}\theta+C=\frac{1}{2}\tan^{-1}\left(\frac{t+1}{2}\right)+C.$

Problem 6. Compute $\int \frac{t^2}{(3+4t-4t^2)^{\frac{3}{2}}}dt$.

$$-4\left[t^2-t-\frac{3}{4}\right]=-4\left[\underbrace{t^2-t+\frac{1}{4}}_{}\underbrace{-\frac{1}{4}-\frac{3}{4}}_{}\right]=$$

$$=-4\left[\left(t-\frac{1}{2}\right)^2-1\right]=4\left[1-\left(t-\frac{1}{2}\right)^2\right]$$

we don't want to keep the "negative 4" because it would make the argument of a square root negative.

$$\int\frac{t^2}{\left\{4\left[1-\left(t-\frac{1}{2}\right)^2\right]\right\}^{3/2}}dt=$$

$$t-\frac{1}{2}=\sin\theta\to t=\frac{1}{2}+\sin\theta$$

$$dt=\cos\theta\,d\theta$$

$$1-\left(t-\frac{1}{2}\right)^2=1-\sin^2\theta=\cos^2\theta$$

$$=\int\frac{\left(\frac{1}{2}+\sin\theta\right)^2}{8\cos^{\cancel{3}2}\theta}\cancel{\cos\theta}\,d\theta=$$

$$=\frac{1}{8}\int\frac{\frac{1}{4}+\sin\theta+\sin^2\theta}{\cos^2\theta}d\theta=$$

$$=\frac{1}{8}\left[\frac{1}{4}\int\frac{d\theta}{\cos^2\theta}+\int\frac{\sin\theta}{\cos^2\theta}d\theta+\int\frac{\sin^2\theta}{\cos^2\theta}d\theta\right]=$$

$$=\frac{1}{8}\left[\frac{1}{4}\int\sec^2\theta\,d\theta+\int\frac{\sin\theta}{\cos^2\theta}d\theta+\int\tan^2\theta\,d\theta\right]=$$

$$\sec^2=1+\tan^2$$

$$\frac{1}{4}\sec^2+\tan^2=$$

$$=\frac{1}{4}\sec^2+\sec^2-1=$$

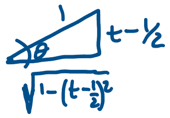
$$=\frac{5}{4}\sec^2-1$$

$$=\frac{1}{8}\left[\int\left(\frac{5}{4}\sec^2\theta-1\right)d\theta+\frac{1}{\cos\theta}\right]=$$

$$=\frac{1}{8}\left[\frac{5}{4}\tan\theta-\theta+\frac{1}{\cos\theta}\right]+C=$$

$$=\frac{1}{8}\left[\frac{5}{4}\frac{t-\frac{1}{2}}{\sqrt{1-\left(t-\frac{1}{2}\right)^2}}-\arcsin\left(t-\frac{1}{2}\right)+\frac{1}{\sqrt{1-\left(t-\frac{1}{2}\right)^2}}\right]+C$$

$$\sin\theta=t-\frac{1}{2}$$



$$\sqrt{1-\left(t^2-t+\frac{1}{4}\right)}$$

$$\sqrt{\frac{3}{4}-t^2+t}$$

Problem 7. Compute $\int \frac{5x+1}{(2x+1)(x-1)} dx$.

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A \quad (x-1)}{2x+1 \quad (x-1)} + \frac{B \quad (2x+1)}{x-1 \quad (2x+1)}$$

$$5x+1 = A(x-1)+B(2x+1)$$

$$5 \quad = A+2B \Rightarrow 5 = A+2A+2 \Rightarrow 3A=3 \Rightarrow A=1$$

$$1 = -A+B \Rightarrow B = A+1 \Rightarrow B = 1+1 = 2$$

$$\int \frac{1}{2x+1} dx + \int \frac{2}{x-1} dx =$$

$$= \frac{1}{2} \ln|2x+1| + \ln|x-1| + C$$

Problem 8. Compute $\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx$.

$$\frac{x^2+x+1}{(x+2)(x+1)^2} = \frac{A}{x+2}\frac{(x+1)^2}{(x+1)^2} + \frac{B}{x+1}\frac{(x+1)(x+2)}{(x+1)(x+2)} + \frac{C}{(x+1)^2}\frac{(x+2)}{(x+2)}$$

$$x^2+x+1 = A(x+1)^2 + B(x+1)(x+2) + C(x+2)$$

$\boxed{x=-1}$ $\cancel{1}\cancel{-1}+1 = 0 \quad + 0 \quad + C(1) \Rightarrow C=1$

$\boxed{x=-2}$ $4-2+1 = A(1) \quad +0 \quad +0 \Rightarrow A=3$

$\boxed{x=0}$ $0+0+1 = A+2B+2C \Rightarrow 1 = 3+2B+2 \Rightarrow 2B = 1-5 = -4$

$B=-2$

$$\int \frac{3}{x+2}dx - \int \frac{2}{x+1}dx + \int \frac{1}{(x+1)^2}dx$$

$\int \frac{1}{y^2}dy = -\frac{1}{y}$

$$3\ln|x+2| - 2\ln|x+1| - \frac{1}{x+1} + C.$$

Problem 9. Compute $\int \frac{10}{(x-1)(x^2+9)} dx$.

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1}\frac{(x^2+9)}{(x^2+9)} + \frac{Bx+C}{x^2+9}\frac{(x-1)}{(x-1)}$$

$$0x^2+0x+10 = A(x^2+9)+(Bx+C)(x-1)$$

$$10 = 9A - C \Rightarrow 10 = 9A+A \Rightarrow A = 1$$

$$\begin{array}{ll} x & 0 = C - B \rightarrow B = C \\ x^2 & 0 = A + B \rightarrow B = -A \end{array}\Bigg\} B = \underline{\underline{C}} = -\underline{\underline{A}}$$

$C = -1$
$B = -1$

$$\int \frac{1}{x-1}dx - \int \frac{x+1}{x^2+9}dx$$

$$\downarrow$$

$$\ln|x-1| - \int \frac{x}{x^2+9}dx - \int \frac{1}{x^2+9}dx$$

$$\ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$u = x^2+9$
$du = 2\boxed{x\,dx}$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Problem 10. Compute $\int \frac{x^5+x-1}{x^3+1}dx$.

$$\begin{array}{r} x^2 \qquad\qquad\qquad\qquad\qquad \\ x^3+1\ \overline{)\,x^5+0x^4+0x^3+0x^2+x-1} \\ \underline{x^5 \qquad\qquad\qquad + x^2 \qquad\quad} \\ // \quad 0x^4+0x^3-x^2+x-1 \end{array}$$

$$\begin{array}{r} 3 \\ 2\,\overline{)\,7} \\ \underline{6} \\ 1 \end{array}$$

$2 \cdot 3 + 1 = 7$

$\Delta = 1 - 4 < 0$

$$\frac{x^5+x-1}{x^3+1} = \frac{\cancel{(x^3+1)}(x^2)}{\cancel{x^3+1}} + \frac{(-x^2+x-1)}{}$$

$$x^3+1 = (x+1)\underbrace{(x^2-x+1)}_{\text{irreducible}}$$

$$\int x^2 + \frac{\overset{(-1)}{\cancel{-x^2+x-1}}}{(x+1)\cancel{(x^2-x+1)}}\,dx$$

$$\int \left(x^2 - \frac{1}{x+1}\right) dx =$$

$$= \frac{x^3}{3} - \ln|x+1| + C$$

$$\begin{array}{r} x^2-x+1 \\ x+1\ \overline{)\,x^3+0x^2+0x+1} \\ \underline{x^3+x^2 \qquad\qquad} \\ // \ -x^2+0x+1 \\ \underline{-x^2-x \qquad} \\ // \quad x+1 \\ \underline{x+1} \\ // \quad // \end{array}$$

Problem 11. Compute $\int \frac{x^3-2x^2+2x-5}{x^4+4x^2+3}\,dx$.

$$x^2 = z$$

$$z^2+4z+3=(z+3)(z+1)$$

$$\frac{x^3-2x^2+2x-5}{(x^2+3)(x^2+1)} = \frac{Ax+B}{x^2+3}\frac{(x^2+1)}{(x^2+1)} + \frac{Cx+D}{x^2+1}\frac{(x^2+3)}{(x^2+3)}$$

$$x^3-2x^2+2x-5=(Ax+B)(x^2+1)+(Cx+D)(x^2+3)$$

$$-5 = B+3D \to B=-5-3D$$

x $$2 = A+3C \to A=2-3C$$

x^2 $$-2 = B+D \to B=-D-2$$

x^3 $$1 = A+C \to A=1-C=1-\frac{1}{2}=\frac{1}{2}=A$$

$$-5-3D=-D-2$$

$$-3 = 2D$$

$$D=-3/2$$

$$1-C=2-3C$$

$$2C=1$$

$$C=\frac{1}{2}$$

$$B=\frac{3}{2}-\frac{4}{2}=-\frac{1}{2}$$

$$\int \frac{\frac{1}{2}x-\frac{1}{2}}{x^2+3}\,dx + \int \frac{\frac{1}{2}x-\frac{3}{2}}{x^2+1}\,dx$$

$$\frac{1}{2}\int\frac{x}{x^2+3}\,dx - \frac{1}{2}\int\frac{1}{x^2+3} + \frac{1}{2}\int\frac{x}{x^2+1}\,dx - \frac{3}{2}\int\frac{1}{x^2+1}\,dx$$

$$\frac{1}{2}\,\frac{1}{2}\ln|x^2+3| - \frac{1}{2}\,\frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + \frac{1}{2}\,\frac{1}{2}\ln|x^2+1| - \frac{3}{2}\arctan x + C.$$

Problem 12. Compute $\int \frac{1}{x\sqrt{x-1}}dx$.

$u = \sqrt{x-1}$

$u^2 = x - 1$

$x = u^2 + 1$

$du = \frac{1}{2\sqrt{x-1}} dx$

$\downarrow$

$dx = 2\sqrt{x-1}\, du =$

$= 2u\, du$

$$\int \frac{1}{(u^2+1)\cancel{u}} 2\cancel{u}\, du$$

$$\int \frac{2}{1+u^2} du =$$

$$= 2 \arctan u + c$$

$$= 2 \arctan \sqrt{x-1} + C$$

Problem 13. Compute $\int \frac{1}{x^2+x\sqrt{x}}dx$.

$u = \sqrt{x}$ $\quad du = \frac{1}{2\sqrt{x}} dx$

$u^2 = x$

$\downarrow$

$u^4 = x^2$

$dx = 2\sqrt{x}\, du =$

$= \underline{2u\, du}$

$$\int \frac{1}{u^4 \;\; + u^3} \cdot 2u\, du$$

$x\sqrt{x} = u^2 \cdot u = u^3$

$$\int \frac{2}{u^3 + u^2} du$$

$u^3 + u^2 = u^2(u+1)$

$$\frac{2}{\underline{\underline{u^2(u+1)}}} = \frac{A}{u}\frac{u(u+1)}{u(u+1)} + \frac{B}{u^2}\frac{(u+1)}{(u+1)} + \frac{C}{u+1}\frac{u^2}{u^2}$$

$$2 = A(u^2+u) + B(u+1) + Cu^2$$

$u=-1 \quad 2 = 0 + 0 + C \Rightarrow C = 2$

$u=0 \quad 2 = 0 + B + 0 \Rightarrow B = 2$

$u=1 \quad 2 = 2A + 2B + C \Rightarrow \cancel{2} = 2A + 4 + \cancel{2} \Rightarrow A = -2$

$$\int \frac{-2}{u} du + \int \frac{2}{u^2} du + \int \frac{2}{u+1} du$$

$$-2 \ln|u| - \frac{2}{u} + 2 \ln|u+1| + C$$

$$-2 \ln|\sqrt{x}| - \frac{2}{\sqrt{x}} + 2 \ln|\sqrt{x}+1| + C$$

Problem 14. Compute $\int \frac{e^{2x}}{e^{2x}+3e^x+2}dx$.

$e^x = u$

$e^{2x} = u^2$

$du = e^x \underline{\underline{dx}}$

$dx = \frac{du}{e^x}$

$$\int \frac{u^2 = e^x \cancel{e^x}}{u^2 + 3u + 2} \frac{du}{\cancel{e^x}}$$

$u^2+3u+2 = (u+2)(u+1)$

$$\int \frac{u}{u^2+3u+2} du$$

$$\frac{u}{(u+2)(u+1)} = \frac{A}{u+2}\frac{(u+1)}{(u+1)} + \frac{B}{u+1}\frac{(u+2)}{(u+2)}$$

$u = A(u+1) + B(u+2)$ $\qquad B = -1$

$1 = A + B \rightarrow A = 1 - B \Big\}$ $1 - B = -2B$ $\qquad A = 2$

$0 = A + 2B \rightarrow A = -2B \Big\}$ $1 = -B$

$$\int \frac{2}{u+2} du - \int \frac{1}{u+1} du$$

$2 \ln|u+2| - \ln|u+1| + C$

$2 \ln|e^x+2| - \ln|e^x+1| + C$.