TEST 1 REVIEW

Problem 1. $\int \frac{\cos^3(\ln x)}{x} dx$ $\qquad u = \ln x \qquad du = \frac{1}{x} dx$

$\int \cos^3(u)\, du$

$\int \cos^2 u \underbrace{\cos u\, du}$

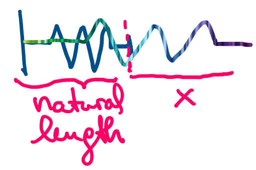
$\int (1-\sin^2 u) \cos u\, du \qquad y = \sin u \quad dy = \cos u\, du$

$\int (1-y^2)\, dy =$

$= y - \frac{1}{3}y^3 + C = \sin(\ln x) - \frac{1}{3}\sin^3(\ln x) + C$

Problem 2. The force required to hold a spring stretched to a length of 7 m is 5 N. Find the work required to stretch the spring from a length of 4 m to 8 m. The natural length of the spring is 3 m.

$$F(x) = kx$$



$$5 = k(7-3)$$

$$5 = k \cdot 4$$

$$k = \frac{5}{4} \qquad F(x) = \frac{5}{4}x$$

$$W = \int_{4-3}^{8-3} \frac{5}{4} x \, dx = \frac{5}{4} \frac{x^2}{2} \Big|_1^5 = \frac{5}{8}(25-1) =$$

$$= \frac{5}{\cancel{8}} \cdot \cancel{24}^{3} = 15 \text{ Nm}$$

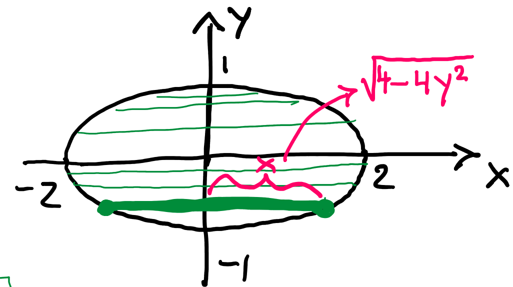
$$15 \text{ J}.$$

Problem 3.

$x^2 = 4 - 4y^2$

Find the volume of the solid S whose base is bounded by the region $x^2 + 4y^2 = 4$, and cross-sections perpendicular to the y-axis are isosceles triangles with height equal to the base.

$$\text{Vol} = \int_{-1}^{1} A(y)\,dy$$

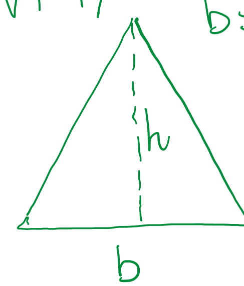


$$A(y) = \frac{1}{2}\left[2\sqrt{4-4y^2}\right]^2 =$$

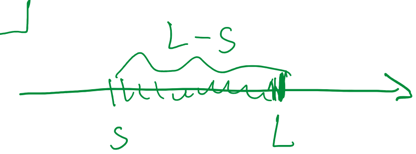
$$= \frac{1}{2}\,4\,(4-4y^2) = \frac{\cancel{16}^{\,8}}{\cancel{2}}(1-y^2)$$

$b = 2\sqrt{4-4y^2}$

$b = h$



$\frac{1}{2}bh = \frac{1}{2}b^2$



$$\text{Vol} = \int_{-1}^{1} 8\,(1-y^2)\,dy =$$

$$= 8\left(y - \frac{1}{3}y^3\right)\Big|_{-1}^{1} =$$

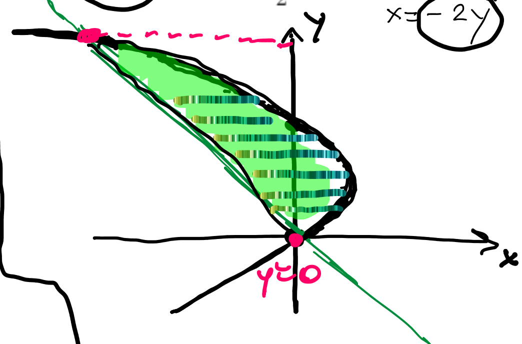
$$= 8\left[1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right] =$$

$$= 8\left(\frac{4}{3}\right) = \frac{32}{3}$$

$y(3-y)=0 \begin{cases} y=0 \\ y=3 \end{cases}$

Problem 4. Find the area bounded by $x = 3y - y^2$ and $y = -\frac{x}{2}$. $\Rightarrow -x = 2y$

$x = -2y$



Area $\int (\text{Right} - \text{Left})\, dy =$

$-2y = 3y - y^2$

$y^2 - 5y = 0$

$y(y-5) = 0 \begin{cases} y=0 \\ y=5 \end{cases}$

$= \int_0^5 \left[(3y-y^2) - (-2y)\right] dy$

$3y - y^2 + 2y = 5y - y^2$

$\int_0^5 (5y - y^2)\, dy$

$\frac{5y^2}{2} - \frac{1}{3}y^3 \Big|_0^5 =$

$= \frac{125}{2} - \frac{125}{3} = 125\left(\frac{3-2}{6}\right) = \frac{125}{6}$

Problem 5. $\int \frac{\ln x}{\sqrt{x}}\,dx$

$u = \ln x$

$du = \frac{1}{x}dx$

$dv = \frac{1}{\sqrt{x}}dx = x^{-1/2}\,dx$

$V = 2\,x^{1/2} = 2\sqrt{x}$

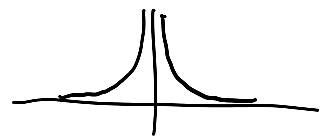
$(\ln x)(2\sqrt{x}) - \int 2x^{1/2}\frac{1}{x}dx$

$\int u\,dv = uv - \int v\,du$

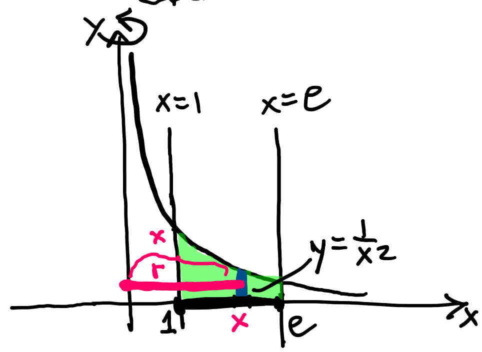
$2\sqrt{x}\ln x - 2\int x^{-1/2}dx =$

$= 2\sqrt{x}\ln x - 2(2\sqrt{x}) + C$

$2\sqrt{x}\ln x - 4\sqrt{x} + C.$



Problem 6. The region bounded by $y = \frac{1}{x^2}$, $x = 1$, $x = e$, and $y = 0$ is rotated around the y-axis. Find the volume.



If you use the washer method, you will have to split this into two integrals.

Let's do cylindrical shells:

height $= \frac{1}{x^2}$

radius $= x$

circumference · height · thickness

$$\int_1^e 2\pi x \cdot \frac{1}{x^2}\,dx = 2\pi \int_1^e \frac{1}{x}\,dx = 2\pi\left[\ln e - \ln 1\right] =$$

$$= 2\pi$$

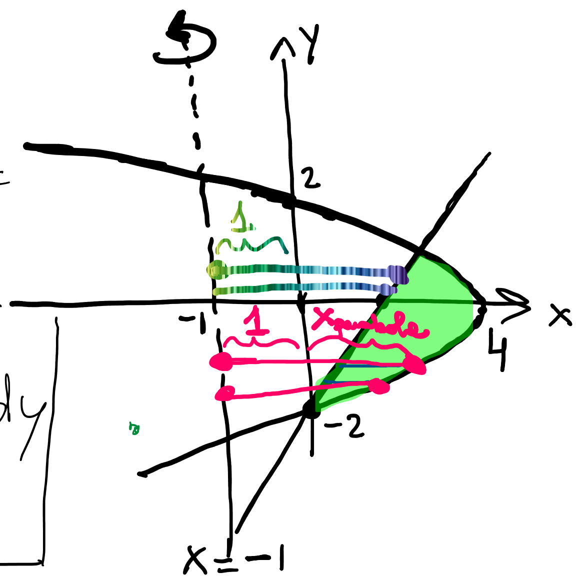
WASHERS

$x-2=y$

Problem 7. The region bounded by $x+y^2=4$ and $x-y=2$ is rotated around the line $x=-1$. Set up but do not evaluate an integral representing the volume of the solid.

$x=4-y^2 \quad , \quad x=2+y$

$$\text{Vol} = \int \pi (R^2 - r^2)\, dy =$$



$$= \boxed{\pi \int (5-y^2)^2 - (3+y)^2 \, dy}$$

$R = 1 + x_{parabola} = 1 + 4 - y^2 =$

$R = 5 - y^2$

$r = 1 + x_{line} = 1 + 2 + y =$

$r = 3 + y$

Problem 8. $\int_0^2 x^2 e^{3x}\, dx$

$u = x^2 \qquad dv = e^{3x}\, dx$

$du = 2x\,dx \qquad v = \frac{1}{3}e^{3x}$

$\int u\,dv = uv - \int v\,du$

$\frac{x^2}{3}e^{3x} - \int \left(\frac{2}{3}\right) x e^{3x}\, dx$

$u = x \qquad dv = e^{3x}$

$du = dx \qquad v = \frac{1}{3}e^{3x}$

$\frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[\frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}\, dx\right] =$

$= \underbrace{\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{9}\frac{1}{3}e^{3x}}\ + C$

now evaluate from 0 to 2.

$\left(\frac{1}{3}4e^6 - \frac{2}{9}2e^6 + \frac{2}{27}e^6\right) - \left(0 - 0 + \frac{2}{27}\cdot 1\right)$

$e^6\left(\frac{4}{3} - \frac{4}{9} + \frac{2}{27}\right) - \frac{2}{27}$

$\frac{36-12+2}{27}$

$\frac{26}{27}e^6 - \frac{2}{27}.$

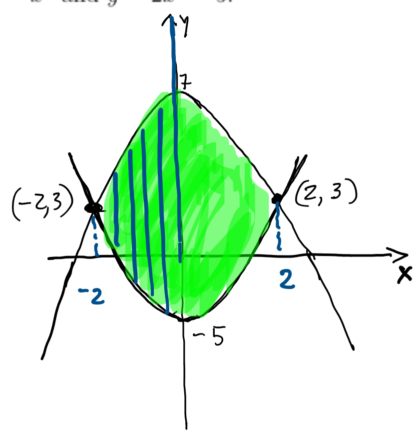
Problem 9. Find the area bounded by $y = 7 - x^2$ and $y = 2x^2 - 5$.

$7 - x^2 = 2x^2 - 5$

$12 = 3x^2$

$x^2 = 4 \Rightarrow x = \pm 2$

$y = 7 - 4 = 3$



Area $= \int \text{Top} - \text{Bottom} \, dx$

$$\int_{-2}^{2} (7 - x^2) - (2x^2 - 5) \, dx$$

$7 - x^2 - 2x^2 + 5$

$$\int_{-2}^{2} 12 - 3x^2 \, dx$$

$$2 \int_{0}^{2} (12 - 3x^2) dx = 2 \left[12x - x^3 \right]_0^2 =$$

$$= 2 \left[24 - 8 - 0 \right] = 32$$

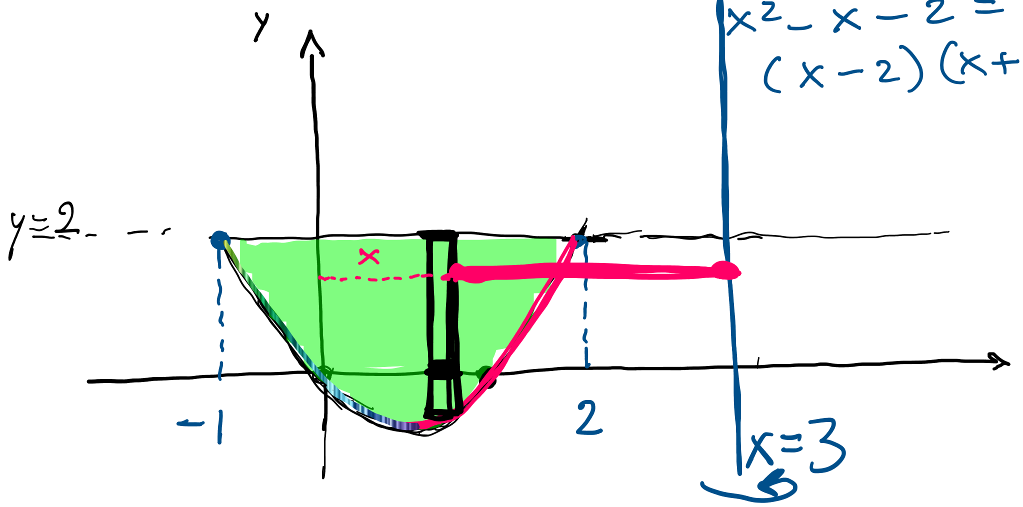
Problem 10. Set up but do not evaluate an integral for the volume of the solid obtained by rotating the region bounded by $y = x^2 - x$ and $y = 2$ rotated around the line $x = 3$.

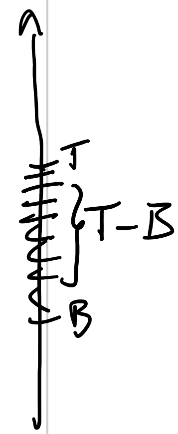
$y = x(x-1)$

$x^2 - x = 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1)$





Cylindrical shells

height $= 2 - (x^2 - x) = 2 - x^2 + x$

radius $= 3 - x$

circumference · height · thickness

$$\int_{-1}^{2} 2\pi(3-x)(2-x^2+x)\,dx$$

Problem 11. $\int \frac{x^3}{(x^2+1)^8}\,dx$

$\underbrace{u = x^2+1}_{x^2 = u-1}$ $du = 2x\,dx$

$x^3 = x^2 \cdot x = (u-1)\,x$

$$\int \frac{(u-1)\,x}{u^8}\,dx = \frac{1}{2}\int \frac{u-1}{u^8}\,du =$$

$$= \frac{1}{2}\int u^{-7} - u^{-8}\,du =$$

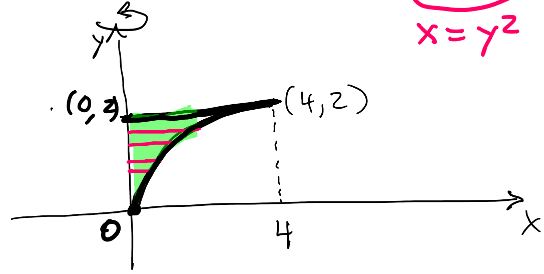
$$= \frac{1}{2}\left[\frac{1}{-6}u^{-6} - \frac{1}{-7}u^{-7}\right] + C =$$

$$= \frac{1}{2}\left[-\frac{1}{6}\frac{1}{(x^2+1)^6} + \frac{1}{7}\frac{1}{(x^2+1)^7}\right] + C$$

Problem 12. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$ around the y-axis.

$x = y^2$

$$\mathrm{Vol} = \int A(y)\,dy$$



radius $= x = y^2$

$$\mathrm{Vol} = \int_0^2 \pi\,\text{radius}^2\,dy =$$

$$= \int_0^2 \pi\,(y^2)^2\,dy = \pi\int_0^2 y^4\,dy = \pi\,\frac{1}{5}y^5\Big|_0^2 =$$

$$= \frac{\pi}{5}(32 - 0) = \frac{32}{5}\pi.$$

Problem 13. $\int \tan^6 x \sec^4 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$\tan^6 x \sec^2 x \sec^2 x$

$$\int \tan^6 x (1 + \tan^2 x) \sec^2 x \, dx$$

$$\int u^6 (1 + u^2) \, du = \int (u^6 + u^8) \, du =$$

$$\frac{u^7}{7} + \frac{u^9}{9} + C = \frac{1}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C.$$

Problem 14. $\int_0^{\pi/6} \sin^2(5x)\,dx$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\int_0^{\pi/6} \frac{1}{2}\left(1-\cos(10x)\right)dx$$

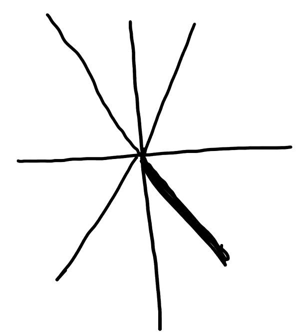
$$\frac{1}{2}\left[x - \frac{1}{10}\sin(10x)\right]_0^{\pi/6} =$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{1}{10}\sin\left(\frac{\overset{5}{\cancel{10}}\pi}{\underset{3}{\cancel{6}}}\right) - 0 + 0\right] =$$

$$= \frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{10}\sin\left(5\frac{\pi}{3}\right)\right) =$$

$$= \frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{10}\left(-\frac{\sqrt{3}}{2}\right)\right) =$$

$$= \frac{1}{2}\left(\frac{\pi}{6} + \frac{\sqrt{3}}{20}\right).$$



Problem 15. Find $\int e^x \sin(8x)\,dx$

$u = e^x \quad dv = \cos(8x)\,dv$

$du = e^x dx \quad v = \frac{1}{8}\sin(8x)$

$u = e^x \quad dv = \sin(8x)\,dx$

$du = e^x dx \quad v = -\frac{1}{8}\cos(8x)$

$$\int e^x \sin 8x\,dx = -\frac{1}{8}e^x\cos(8x) \overset{+}{-} \int -\frac{1}{8}e^x \cos(8x)\,dx =$$

$$\int e^x \sin(8x)\,dx = -\frac{1}{8}e^x\cos(8x) + \frac{1}{8}\left[\frac{e^x}{8}\sin(8x) - \int \frac{1}{8}e^x \sin(8x)\,dx\right]$$

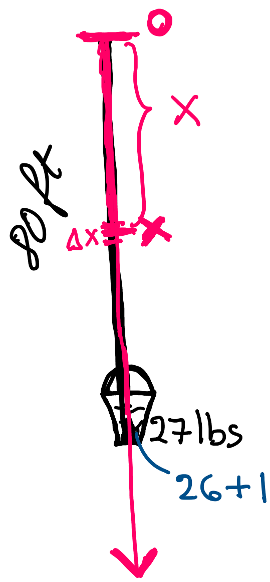
$$\underline{\int e^x \sin 8x\,dx} = -\frac{e^x}{8}\cos 8x + \frac{e^x}{64}\sin 8x \underline{- \frac{1}{64}\int e^x \sin 8x\,dx}$$

$$\frac{65}{64}\int e^x \sin 8x\,dx = -\frac{e^x}{8}\cos 8x + \frac{e^x}{64}\sin 8x$$

$$\int e^x \sin 8x\,dx = \frac{64}{65}\left[-\frac{e^x}{8}\cos 8x + \frac{e^x}{64}\sin 8x\right] + C \qquad 1 + \frac{1}{64} = \frac{64+1}{64}$$

$$\frac{1}{65}\left[-8e^x\cos 8x + e^x \sin 8x\right] + C$$

Problem 16. A bucket attached to a 20 pound rope is used to draw water out of an 80 ft well. The bucket weighs 1 pound and holds 26 pounds of water. How much work is done in drawing up one full bucket of water?



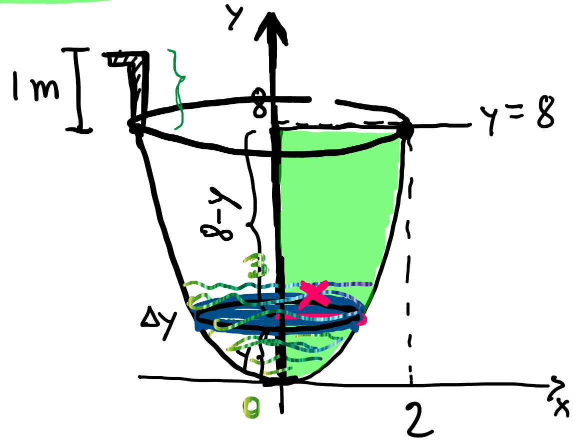
$$\text{density} = \frac{\cancel{20}^{\,1}\ \text{lb}}{{}_4\cancel{80}\ \text{ft}} = \frac{1}{4}\ \text{lb/ft}$$

$$\left.\begin{array}{l}\Delta F = \frac{1}{4}\Delta x \\ \text{distance} = x\end{array}\right\} \Delta W = \left(\frac{1}{4}\Delta x\right)(x)$$

$$W_{rope} = \int_0^{80} \frac{1}{4} x\, dx = \frac{1}{4}\frac{x^2}{2}\Big|_0^{80} = \frac{1}{\cancel{8}}\left(\cancel{80}^{\,10} \cdot 80 - 0\right)$$

$$W_{TOT} = W_{Bucket} + W_{rope} = (27 \cdot 80 + 800)\ \text{ft-lb}$$

Problem 17. Consider the region R bounded by $y = x^3$, $y = 8$, and $x = 0$. Suppose a tank is in the shape of the region R revolved around the y-axis, and the units are measured in meters. If the tank is filled with water to a depth of 3 m, set up but do not evaluate an integral that gives the work done in pumping all the water out of a 1 m high spout. Use ρg for the weight density of water.



$$W = \int F(y) \cdot \text{distance}$$

$$\Delta F = \rho g \, \Delta V$$

$$\Delta V = \pi \, \text{radius}^2 \, \Delta y = \pi y^{2/3} \Delta y$$

$$\Delta F = \rho g \pi y^{2/3} \Delta y$$

$$y = x^3$$

$$y^{1/3} = \underbrace{x = \sqrt[3]{y}}_{\text{radius}}$$

$$\text{distance} = (8-y) + 1 = 9 - y$$

$$W = \int_0^3 \rho g \pi y^{2/3} (9-y) \, dy \quad J.$$

Continue work here.