NOTE ♯2: VOLUMES (DISKS AND WASHERS)

Problem 1. Find the volume of the solid obtained by rotating the region bounded by $y = -x^2+1$, $y = 0$; $x = 0$; about the x axis.

x-axis y-axis

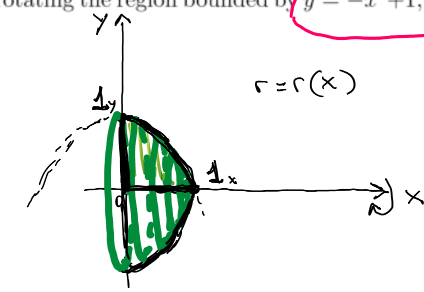
$$\text{Vol} = \int_0^{1_x} A(x)\, dx =$$

$$= \int_0^1 \pi (1-x^2)^2\, dx =$$

$$= \pi \int_0^1 (1 - 2x^2 + x^4)\, dx =$$

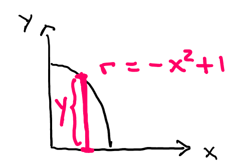
$$= \pi \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1 =$$

$$= \pi \left(1 - \frac{2}{3} - \frac{1}{5}\right) = \pi \frac{15 - 10 - 3}{15} = \frac{2}{15}\pi$$



$$A(x) = \pi r^2$$

$$A(x) = \pi [r(x)]^2$$

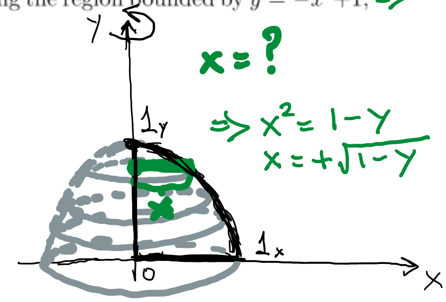


Problem 2. Find the volume of the solid obtained by rotating the region bounded by $y = -x^2+1$, $\Rightarrow$ $y = 0$; $x = 0$; about the y axis.

$x = ?$

$\Rightarrow x^2 = 1-y$

$x = +\sqrt{1-y}$



$r = r(y) = \sqrt{1-y}$

$$\text{VOL} = \int_0^{1y} A(y)\, dy =$$

$$= \int_0^1 \pi \left(\sqrt{1-y}\right)^2 dy =$$

$$= \pi \int_0^1 (1-y)\, dy =$$

$$= \pi \left[y - \frac{1}{2} y^2\right]_0^1 = \pi \left(1 - \frac{1}{2}\right) = \frac{\pi}{2}.$$

Problem 3. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{\ln x}$, $y = 0$; $x = e$; about the x axis (you may use the fact that $\int \ln x dx = x \ln x - x + C$).

$\underline{\underline{dx}}$

$$Vol = \int_1^e A(x)\,dx = \int_1^e \pi (r(x))^2\,dx =$$

$$= \int_1^e \pi \left(\sqrt{\ln x}\right)^2 dx = \pi \int_1^e \ln x\,dx =$$

$\ln 1 = 0$

$\ln e = 1$

$\log_b b = 1$

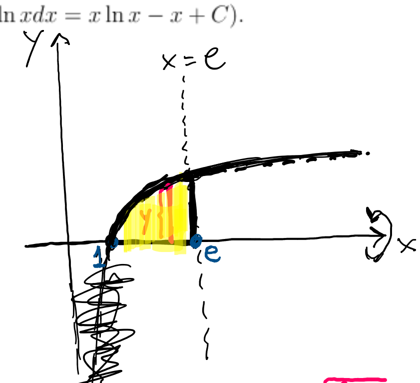
$$= \pi \left[x \ln x - x \right]_1^e =$$

$$= \pi \left[e \ln e - e - (1 \ln 1 - 1) \right] =$$

$$= \pi \left(\cancel{e} - \cancel{e} - (0 - 1) \right) = \boxed{\pi}$$

BY PARTS

$$\boxed{\int \ln x\,dx = x \ln x - x + C}$$



$r = r(x) = y = \sqrt{\ln x}$

$\int (\ln x)^2\,dx$

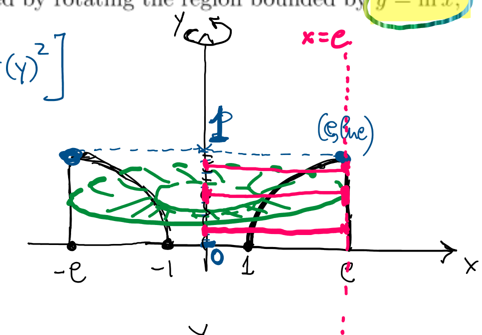
dy

Problem 4. Find the volume of the solid obtained by rotating the region bounded by $y = \ln x$, $y = 0$; $x = e$; about the y axis.

$$A(y) = \pi[R(y)]^2 - \pi[r(y)]^2 = \pi\left[R(y)^2 - r(y)^2\right]$$

$R(y) = e$

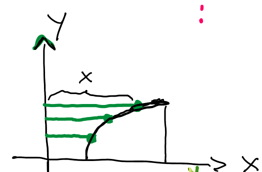
$r(y) = e^y$



$$\text{Vol} = \int_0^1 A(y)\,dy =$$

$$= \int_0^1 \pi\left[e^2 - (e^y)^2\right] dy =$$

$$(e^y)^2 = e^y \cdot e^y = e^{2y}$$



$$= \pi \int_0^1 \left(e^2 - e^{2y}\right) dy = \pi\left[e^2 y - \frac{1}{2} e^{2y}\right]_0^1 =$$

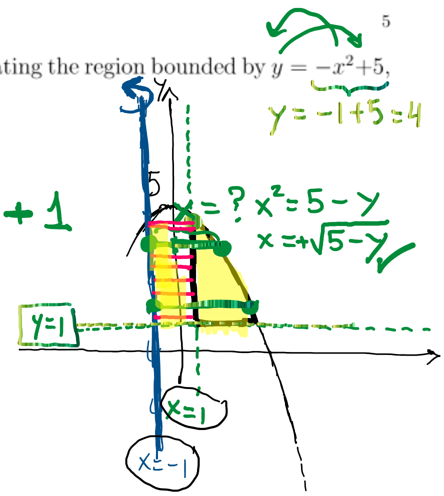
if $y = \ln x \Rightarrow x = e^y$

$$= \pi\left[e^2 - \frac{e^2}{2} - \left(0 - \frac{1}{2}e^0\right)\right] =$$

$$\boxed{\int e^{2y}\,dy \quad u = 2y \quad \ldots}$$

$$= \pi\left(\frac{e^2}{2} + \frac{1}{2}\right) = \pi\,\frac{1 + e^2}{2} \quad \checkmark$$

Problem 5. Find the volume of the solid obtained by rotating the region bounded by $y = -x^2+5$, $y = 1$; $x = 1$; about the line $x = -1$.



$$\text{VOL} = \int_1^4 A(y)\,dy = \quad * R = \sqrt{5-y} + 1 \quad r = 2$$

dy

$$= \int_1^4 \pi\left[R_y^2 - r_y^2\right]dy =$$

$$= \int_1^4 \pi\left[\left(\sqrt{5-y}+1\right)^2 - (2)^2\right]dy =$$

$$5 - y + 2\sqrt{5-y} + 1 - 4 =$$

$$2 - y + 2\sqrt{5-y}$$

2 units

−1 0 1

$$= \pi\int_1^4 \left(2 - y + 2\sqrt{5-y}\right)dy =$$

$$= \pi\left[2y - \frac{1}{2}y^2 - \frac{4}{3}(5-y)^{3/2}\right]_1^4 =$$

$u = 5-y \quad du = -dy$

$$\int -2u^{1/2}\,du = -2\,\frac{2}{3}u^{3/2} = -\frac{4}{3}(5-y)^{3/2}$$

$$= \pi\left[8 - \frac{\cancel{16}\,8}{\cancel{2}} - \frac{4}{3}\cdot 1^{3/2} - \left(2 - \frac{1}{2} - \frac{4}{3}\cdot 4^{3/2}\right)\right] =$$

$$= \pi\left(-\frac{4}{3} - 2 + \frac{1}{2} + \frac{4}{3}\cdot 8\right) =$$

$$= \pi\left(\frac{47}{6}\right) = \boxed{\frac{47}{6}\pi}$$

$$+ \frac{3}{2}\,\frac{2}{\cancel{4}}\,\frac{}{\cancel{3}}(5-y)^{1/2}$$

$$-\frac{4}{3} - 2 + \frac{1}{2} + \frac{32}{3} = \frac{-8-12+3+64}{6} = \frac{47}{6}$$

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1 2 2

$$-\frac{4}{3}-2+\frac{1}{2}+\frac{32}{3}=\frac{-8-12+3+64}{6}=$$

$$=\frac{47}{6}$$

Problem 6. Find the volume of the solid obtained by rotating the region bounded by $y=-x^2+5$, $y=1$; $x=1$; about the line $y=-1$.

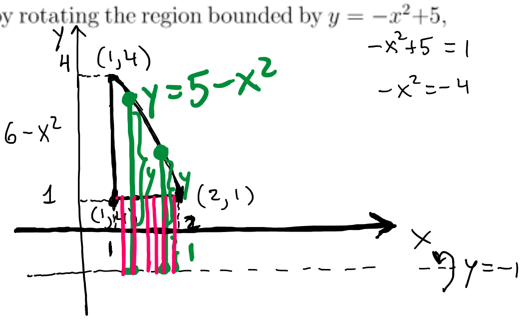
$-x^2+5=1$

$-x^2=-4$

$\cancel{dx}$

$R=(5-x^2)+1=6-x^2$

$r=2$



$$\text{VOL}=\int_1^2 A(x)\,dx=$$

$$=\int_1^2 \pi\left[(6-x^2)^2-(2)^2\right]dx=$$

$$=\pi\int_1^2 (36-12x^2+x^4-4)\,dx=$$

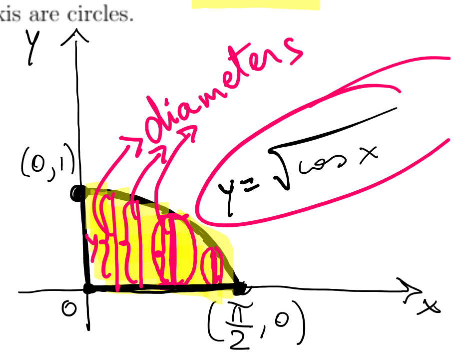
$$=\pi\int_1^2 (x^4-12x^2+32)\,dx=\pi\left[\frac{x^5}{5}-\cancel{12}^{\,4}\frac{x^3}{\cancel{3}}+32x\right]_1^2=$$

$$=\pi\left(\frac{32}{5}-\cancel{32}+\cancel{64}-\frac{1}{5}+4-\cancel{32}\right)=\pi\left(\frac{31+20}{5}\right)=$$

$$=\frac{51}{5}\pi \checkmark$$

dx

Problem 7. Find the volume of the solid whose base is the region bounded by $y = \sqrt{\cos x}$, $y = 0$, $x = 0$ and whose cross sections perpendicular to the x axis are circles.



$$\text{VOL} = \int_0^{\pi/2} A(x)\,dx =$$

$$= \int_0^{\pi/2} \frac{\pi}{4}\cos x\,dx =$$

$$= \frac{\pi}{4}\sin x\Big|_0^{\pi/2} = \frac{\pi}{4}(1-0) =$$

$$= \frac{\pi}{4}$$

$$A(x) = \pi\,(r(x))^2$$

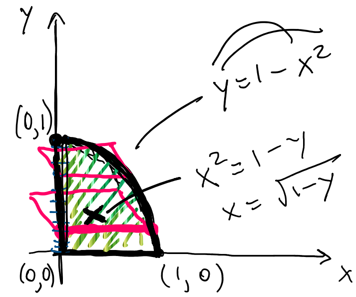
$$\text{diameter} = y = \sqrt{\cos x}$$

$$\text{radius} = \frac{1}{2}\sqrt{\cos x}$$

$$A(x) = \pi\,\frac{1}{4}\cos x$$

Problem 8. Find the volume of the solid whose base is the region bounded by $y = -x^2 + 1$, $y = 0$, $x = 0$ and whose cross sections perpendicular to the y axis are squares.

dy



$$A(y) = (\text{side})^2 = \left(\sqrt{1-y}\right)^2$$

$$\text{VOL} = \int_0^1 A(y)\,dy =$$

$$= \int_0^1 (1-y)\,dy = y - \frac{1}{2}y^2 \Big|_0^1 =$$

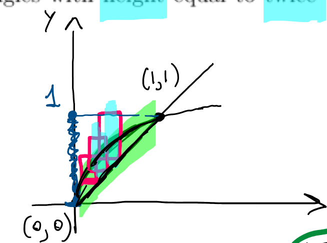
$$= 1 - \frac{1}{2} - 0 = \frac{1}{2} \text{ units}^3$$

Problem 9. Find the volume of the solid whose base is the region bounded by $\underbrace{y = \sqrt{x}}_{y^2 = x}$, $y = x$ and whose cross sections perpendicular to the y axis are rectangles with height equal to twice the base.

dy

$2b = h$

$b = y - y^2$



$$Vol = \int_0^1 A(y)\,dy =$$

$$= \int_0^1 2\left(y - y^2\right)^2 dy =$$

$$= 2\int_0^1 y^2 - 2y^3 + y^4 \, dy =$$

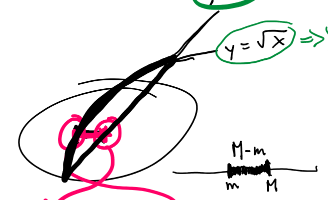
$$= 2\left[\frac{1}{3}y^3 - \frac{2}{4}y^4 + \frac{1}{5}y^5\right]_0^1 =$$

$$= 2\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = 2\left(\frac{10 - 15 + 6}{30}\right) =$$

$$= 2 \cdot \frac{1}{30} = \frac{1}{15}.$$

$y = x \Rightarrow x = y$

$y = \sqrt{x} \Rightarrow y^2 = x$



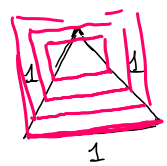
$b = x_{line} - x_{square\ root} =$

$b = y - y^2$

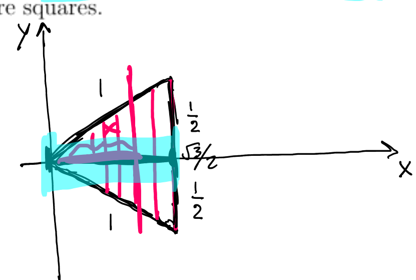
$A(y) = base * height =$

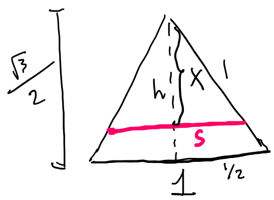
$= base * 2\,base = 2\,base^2$

Problem 10. Find the volume of the solid whose base is an equilateral triangle of side length one and whose cross sections parallel to one of the sides are squares.



$$h = \sqrt{1^2 - \left(\tfrac{1}{2}\right)^2} = \sqrt{1 - \tfrac{1}{4}} = \sqrt{\tfrac{3}{4}} = \tfrac{1}{2}\sqrt{3}.$$





$$\frac{s}{1} = \frac{x}{\frac{\sqrt{3}}{2}}$$

$$\frac{s}{1} = \boxed{\frac{2x}{\sqrt{3}}}$$

$$\text{VOL} = \int_0^{\sqrt{3}/2} (\text{side})^2\, dx = \int_0^{\sqrt{3}/2} \left(\frac{2x}{\sqrt{3}}\right)^2 dx = \int_0^{\sqrt{3}/2} \frac{4}{3}x^2\, dx =$$

$$= \frac{4}{3}\, \frac{x^3}{3}\Bigg|_0^{\sqrt{3}/2} = \frac{4}{9}\left[\left(\frac{\sqrt{3}}{2}\right)^3 - 0^3\right] =$$

$$= \frac{\cancel{4}^{1}}{\cancel{9}_{3}}\, \frac{\cancel{8}^{1}\sqrt{3}}{\cancel{8}_{2}} = \frac{\sqrt{3}}{6} = \frac{1}{6}\sqrt{3}$$